# Reflection about a geodesic passing through two given points in the Poincaré and Klein disk models of hyperbolic geometry

Frank Nielsen
Sony Computer Science Laboratories Inc
Tokyo

15 December 2023

**Abstract**

We report in this note a direct closed-form formula for the reflection operation about a geodesic defined by two given points in the Poincaré disk.

## 1 Geodesics and pregeodesics

In metric geometry (including Riemannian geometry) with distance $\rho$, a geodesic $\gamma(l)$ is parameterized by a constant speed parameter $l$ so that we have

$$\rho(\gamma(l), \gamma(l')) = |l - l'| \, \rho(\gamma(0), \gamma(1)).$$

The constant speed parameterization is related to arc length parameterization by $s = l\, \rho(\gamma(0), \gamma(1))$, where $p_0 = \gamma(0)$ and $p_1 = \gamma(1)$ are the geodesic endpoints.

To constrast with geodesics, a pregeodesic $\bar{\gamma}(t) = \bar{\gamma}(l(t))$ is a reparameterization of the geodesic such that $l(t)$ is a smooth and invertible with inverse function $t(l)$. Pregeodesics can yield simplified mathematical expressions of parameterizations, and express equivalently the geodesic curves:

$$c_\gamma = \{\gamma(l) : l \in [0,1]\} = \{\bar{\gamma}(t) : t \in [a,b]\}.$$

For example, in the Klein model of hyperbolic geometry, pregeodesics passing through two given points $k_1$ and $k_2$ of the unit disk $\mathbb{D}$ are Euclidean line segments parameterized by:

$$\bar{\gamma}_K(t) = k_1 + t(k_2 - k_1),$$

with geodesic curve $c_{\gamma_K} = [k_1 k_2]$. The geodesic equation with constant speed parameterization in Klein model has been reported in [4], i.e., the function $l(t)$ such that

$$\gamma_K(l) = \bar{\gamma}_K(l(t))$$

is given in closed-form.

## 2 Geodesic curves in Klein model

To find the complete geodesic $\Gamma_K = \{\gamma_K(l) : l \in \mathbb{R}\}$ in Klein model, we need to find $t_m$ and $t_M$ such that $\Gamma_K$ is the line passing through $[k_1 k_2]$ clipped to the unit disk $\mathbb{D}$. $t_m$ and $t_M$ are the two solutions of the quadratic equation:

$$\langle k_1 + t(k_2 - k_1), k_1 + t(k_2 - k_1)\rangle = 1$$

A model of a geometry is said conformal if the angles of two curves $c_1(t)$ and $c_2(t)$ intersecting at $t_0$ match the Euclidean angles. The Poincaré disk model is conformal but not the Klein model (except at the origin).

$$
\begin{aligned}
&[a0=-\frac{(b2-b1)\,b3^2+(-b2^2+b1^2-a2^2+a1^2)\,b3+b1\,b2^2+(-b1^2+a3^2-a1^2)\,b2+(a2^2-a3^2)\,b1}{(2\,a2-2\,a1)\,b3+(2\,a1-2\,a3)\,b2+(2\,a3-2\,a2)\,b1},\ b0=\\
&\frac{(a2-a1)\,b3^2+(a1-a3)\,b2^2+(a3-a2)\,b1^2+(a2-a1)\,a3^2+(a1^2-a2^2)\,a3+a1\,a2^2-a1^2\,a2}{(2\,a2-2\,a1)\,b3+(2\,a1-2\,a3)\,b2+(2\,a3-2\,a2)\,b1},\ r0=\mathrm{sqrt}((b2^2-2\,b1\,b2+b1^2+a2^2-2\,a1\,a2+a1^2)\,b3^4+\\
&(-2\,b2^3+2\,b1\,b2^2+(2\,b1^2-2\,a2^2+4\,a1\,a2-2\,a1^2)\,b2-2\,b1^3+(-2\,a2^2+4\,a1\,a2-2\,a1^2)\,b1)\,b3^3+(b2^4+2\,b1\,b2^3+(-6\,b1^2+2\,a3^2+(-2\,a2-2\,a1)\,a3+2\,a2^2-2\,a1\,a2+2\,a1^2)\,b2^2+\\
&\left(2\,b1^3+(-4\,a3^2+(4\,a2+4\,a1)\,a3+2\,a2^2-8\,a1\,a2+2\,a1^2)\,b1\right)b2+b1^4+(2\,a3^2+(-2\,a2-2\,a1)\,a3+2\,a2^2-2\,a1\,a2+2\,a1^2)\,b1^2+(2\,a2^2-4\,a1\,a2+2\,a1^2)\,a3^2+\\
&(-2\,a2^3+2\,a1\,a2^2+2\,a1^2\,a2-2\,a1^3)\,a3+a2^4-2\,a1\,a2^3+2\,a1^2\,a2^2-2\,a1^3\,a2+a1^4)\,b3^2+(-2\,b1\,b2^4+(2\,b1^2-2\,a3^2+4\,a1\,a3-2\,a1^2)\,b2^3+\\
&\left(2\,b1^3+(2\,a3^2+(4\,a2-8\,a1)\,a3-4\,a2^2+4\,a1\,a2+2\,a1^2)\,b1\right)b2^2+\\
&\left(-2\,b1^4+(2\,a3^2+(4\,a1-8\,a2)\,a3+2\,a2^2+4\,a1\,a2-4\,a1^2)\,b1^2+(-2\,a2^2+4\,a1\,a2-2\,a1^2)\,a3^2+(4\,a1\,a2^2-8\,a1^2\,a2+4\,a1^3)\,a3-2\,a1^2\,a2^2+4\,a1^3\,a2-2\,a1^4\right)b2+\\
&(-2\,a3^2+4\,a2\,a3-2\,a2^2)\,b1^3+((-2\,a2^2+4\,a1\,a2-2\,a1^2)\,a3^2+(4\,a2^3-8\,a1\,a2^2+4\,a1^2\,a2)\,a3-2\,a2^4+4\,a1\,a2^3-2\,a1^2\,a2^2)\,b1)\,b3+(b1^2+a3^2-2\,a1\,a3+a1^2)\,b2^4+\\
&((-2\,a3^2+4\,a1\,a3-2\,a1^2)\,b1-2\,b1^3)\,b2^3+\\
&\left(b1^4+(2\,a3^2+(-2\,a2-2\,a1)\,a3+2\,a2^2-2\,a1\,a2+2\,a1^2)\,b1^2+a3^4+(-2\,a2-2\,a1)\,a3^3+(2\,a2^2+2\,a1\,a2+2\,a1^2)\,a3^2+(-4\,a1\,a2^2+2\,a1^2\,a2-2\,a1^3)\,a3+2\,a1^2\,a2^2-2\,a1^3\,a2+a1^4\right)\\
&b2^2+((-2\,a3^2+4\,a2\,a3-2\,a2^2)\,b1^3+(-2\,a3^4+(4\,a2+4\,a1)\,a3^3+(-2\,a2^2-8\,a1\,a2-2\,a1^2)\,a3^2+(4\,a1\,a2^2+4\,a1^2\,a2)\,a3-2\,a1^2\,a2^2)\,b1)\,b2+(a3^2-2\,a2\,a3+a2^2)\,b1^4+\\
&(a3^4+(-2\,a2-2\,a1)\,a3^3+(2\,a2^2+2\,a1\,a2+2\,a1^2)\,a3^2+(-2\,a2^3+2\,a1\,a2^2-4\,a1^2\,a2)\,a3+a2^4-2\,a1\,a2^3+2\,a1^2\,a2^2)\,b1^2+(a2^2-2\,a1\,a2+a1^2)\,a3^4+\\
&(-2\,a2^3+2\,a1\,a2^2+2\,a1^2\,a2-2\,a1^3)\,a3^3+(a2^4+2\,a1\,a2^3-6\,a1^2\,a2^2+2\,a1^3\,a2+a1^4)\,a3^2+(-2\,a1\,a2^4+2\,a1^2\,a2^3+2\,a1^3\,a2^2-2\,a1^4\,a2)\,a3+a1^2\,a2^4-2\,a1^3\,a2^3+a1^4\,a2^2)/\\
&((2\,a2-2\,a1)\,b3+(2\,a1-2\,a3)\,b2+(2\,a3-2\,a2)\,b1)]
\end{aligned}
$$

Figure 1: Solution of the systems of equation defined in Eq. 1.

# 3 Reflections about geodesics in the Poincaré disk model

Geodesics in the Poincaré model are either arc of circles perpendicular to the disk boundary $\partial\mathbb{D}$, or line segments passing through the origin. Poincaré geodesic curves are sometimes called clines [1] (circles or lines). The isometries in the Poincaré model seen as a subset of the complex plane $\mathbb{C}$ are Möbius transformations. In particular, reflections around a complete geodesic curve $\Gamma_{z_1 z_2}$ passing through two given points $z_1, z_2 \in \mathbb{D}$ is an Euclidean circle inversion, and thus defined by

$$\mathrm{reflect}(z) = \frac{r_0^2}{\bar{z} - \bar{z}_0} + z_0.$$

For example, such reflection operations are used for embeddings tree structures into the Poincaré disk [5]. Usually, reflections are implemented via translations to the origin of the disk. We report in this note a direct closed-form formula.

The reflection leaves the geodesic $\Gamma_{z_1 z_2}$ invariant:

$$\forall z \in \Gamma_{z_1 z_2}, \mathrm{reflect}(z) = z,$$

and we have $\mathrm{reflect}(\mathrm{reflect}(z)) = z$ for any $z \in \mathbb{D}$.

Let $z_3 \in \Gamma_{z_1 z_2}$ be any point inside the Poincaré geodesic. For example, we consider $k_3$ the Klein interior point given by $\frac{k_1+k_2}{2}$ where

$$k_i = \frac{2}{1 + z_i \bar{z}_i}\, z_i, \quad i \in \{1, 2\},$$

and we convert the Klein geodesic interior point to the Poincaré disk:

$$z_3 = \frac{1 - \sqrt{1 - \|k_3\|^2}}{\|k_3\|^2}\, k_3$$

Then we write the system of equations with $z_j = a_j + ib_j$ for $j \in \{0, 1, 2, 3\}$:

$$\left\{ \begin{array}{lcl} \mathrm{reflect}(a_1 + ib_1) & = & a_1 + ib_1, \\ \mathrm{reflect}(a_2 + ib_2) & = & a_2 + ib_2, \\ \mathrm{reflect}(a_3 + ib_3) & = & a_3 + ib_3. \end{array} \right. \tag{1}$$

Next, we use the computer algebra system `Maxima`[1] to solve the system of Eq. 1 (Figure 1) using the

[1]`https://maxima.sourceforge.io/`

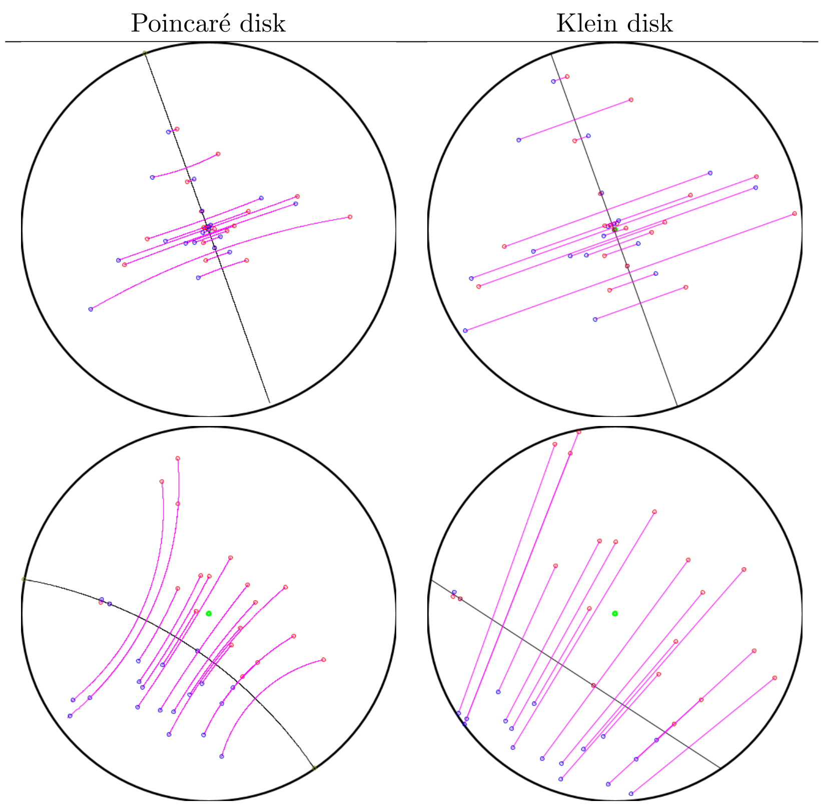


Figure 2: Illustrating reflections around geodesics on the Poincaré and Klein disks.

code:

```
z1:a1+b1*%i;
z2:a2+b2*%i;
z3:a3+b3*%i;

/* circle inversion */
reflect(z,a0,b0,r0):=(r0**2/(conjugate(z)-conjugate(a0+b0*%i)))+a0+b0*%i;

/* reflection of the points z1, z2, z3 should be identity */
reflect(z1,a0,b0,r0);
realpart(%);ratsimp(%);
eq1: %=a1;
reflect(z1,a0,b0,r0);
imagpart(%);ratsimp(%);
eq2: %=b1;
 reflect(z2,a0,b0,r0);
realpart(%);ratsimp(%);
eq3: %=a2;
reflect(z2,a0,b0,r0);
imagpart(%);ratsimp(%);
eq4: %=b2;
 reflect(z3,a0,b0,r0);
realpart(%);ratsimp(%);
eq5: %=a3;
reflect(z3,a0,b0,r0);
imagpart(%);ratsimp(%);
eq6: %=b3;

solve([eq1,eq2,eq3,eq4,eq5,eq6],[a0,b0,r0]);
solution: %[2];
```

`Maxima` reports the unique solution for $z_0 = a_0 + ib_0$ and $r_0$ defining the reflection:

$$a_0 = -\frac{(b_2 - b_1)\, b_3^2 + \left(-b_2^2 + b_1^2 - a_2^2 + a_1^2\right)\, b_3 + b_1\, b_2^2 + \left(-b_1^2 + a_3^2 - a_1^2\right)\, b_2 + \left(a_2^2 - a_3^2\right)\, b_1}{(2\, a_2 - 2\, a_1)\, b_3 + (2\, a_1 - 2\, a_3)\, b_2 + (2\, a_3 - 2\, a_2)\, b_1}$$

$$b_0 = \frac{(a_2 - a_1)\, b_3^2 + (a_1 - a_3)\, b_2^2 + (a_3 - a_2)\, b_1^2 + (a_2 - a_1)\, a_3^2 + \left(a_1^2 - a_2^2\right)\, a_3 + a_1\, a_2^2 - a_1^2\, a_2}{(2\, a_2 - 2\, a_1)\, b_3 + (2\, a_1 - 2\, a_3)\, b_2 + (2\, a_3 - 2\, a_2)\, b_1}$$

Parameter $r_0$ has a complex expression reported in the screenshot 1.
A point $z = a + ib$ is reflected to a point $z' = a' + ib'$ with

$$\begin{aligned} a' &= \frac{(a - a_0)r_0^2}{(b_0 - b)^2 + (a - a_0)^2)} + a_0, \\ b' &= \frac{(b - b_0)r_0^2}{(b_0 - b)^2 + (a - a_0)^2)} + b_0. \end{aligned}$$

Figure 2 illustrates some examples of reflections obtained in the Poincaré disk and then converted into the Klein disk.

Notice that the equation of a circle of center $(x_0, y_0)$ and radius $r_0 > 0$ passing through three given points $(x_1, y_1), (x_2, y_2)$ and $(x_3, y3)$ can be found also by symbolic computing:

```
/* equation of a circle with center (x0,y0) and radius r0 passing through 3 points (xi, yi)
assume(r0>0);

eq1: (x1-x0)**2 + (y1-y0)**2= r0**2;
eq2: (x2-x0)**2 + (y2-y0)**2 = r0**2;
eq3: (x3-x0)**2 + (y3-y0)**2 = r0**2;

solve([eq1,eq2,eq3],[x0,y0,r0]);
%[2];
```

See Figure 3 for a screenshot of Maxima result. One way to geometrically derive this equation is to consider the unique intersection of two line bisectors defined by the three points $\{(x_i, y_i), i \in \{1, 2, 3\}\}$, and then calculate the radius $r_0$ as the distance of the circumcenter to the given points [2].

It is usually easy and numerically robust to implement geometric algorithms like the Voronoi diagram [3] in the Klein disk and visualize the result in the conformal Poincaré disk. The Poincaré disk may be directly considered for some problems like calculating the smallest enclosing hyperbolic ball since a hyperbolic ball in the Poincaré disk model is a Euclidean ball with a shifted center [6].

$$
\begin{aligned}
[\, x_0 = & -\frac{(y_2-y_1)\,y_3^2+(-y_2^2+y_1^2-x_2^2+x_1^2)\,y_3+y_1\,y_2^2+(-y_1^2+x_3^2-x_1^2)\,y_2+(x_2^2-x_3^2)\,y_1}{(2\,x_2-2\,x_1)\,y_3+(2\,x_1-2\,x_3)\,y_2+(2\,x_3-2\,x_2)\,y_1},\ y_0 = \\
& \frac{(x_2-x_1)\,y_3^2+(x_1-x_3)\,y_2^2+(x_3-x_2)\,y_1^2+(x_2-x_1)\,x_3^2+(x_1^2-x_2^2)\,x_3+x_1\,x_2^2-x_1^2\,x_2}{(2\,x_2-2\,x_1)\,y_3+(2\,x_1-2\,x_3)\,y_2+(2\,x_3-2\,x_2)\,y_1},\ r_0 = \mathrm{sqrt}(\,(y_2^2-2\,y_1\,y_2+y_1^2+x_2^2-2\,x_1\,x_2+x_1^2)\,y_3^4+ \\
& (-2\,y_2^3+2\,y_1\,y_2^2+(2\,y_1^2-2\,x_2^2+4\,x_1\,x_2-2\,x_1^2)\,y_2-2\,y_1^3+(-2\,x_2^2+4\,x_1\,x_2-2\,x_1^2)\,y_1)\,y_3^3+(y_2^4+2\,y_1\,y_2^3+(-6\,y_1^2+2\,x_3^2+(-2\,x_2-2\,x_1)\,x_3+2\,x_2^2-2\,x_1\,x_2+2\,x_1^2)\,y_2^2+ \\
& \left(2\,y_1^3+(-4\,x_3^2+(4\,x_2+4\,x_1)\,x_3+2\,x_2^2-8\,x_1\,x_2+2\,x_1^2)\,y_1\right)y_2+y_1^4+(2\,x_3^2+(-2\,x_2-2\,x_1)\,x_3+2\,x_2^2-2\,x_1\,x_2+2\,x_1^2)\,y_1^2+(2\,x_2^2-4\,x_1\,x_2+2\,x_1^2)\,x_3^2+ \\
& (-2\,x_2^3+2\,x_1\,x_2^2+2\,x_1^2\,x_2-2\,x_1^3)\,x_3+x_2^4-2\,x_1\,x_2^3+2\,x_1^2\,x_2^2-2\,x_1^3\,x_2+x_1^4)\,y_3^2+(-2\,y_1\,y_2^4+(2\,y_1^2-2\,x_3^2+4\,x_1\,x_3-2\,x_1^2)\,y_2^3+ \\
& \left(2\,y_1^3+(2\,x_3^2+(4\,x_2-8\,x_1)\,x_3-4\,x_2^2+4\,x_1\,x_2+2\,x_1^2)\,y_1\right)y_2^2+ \\
& \left(-2\,y_1^4+(2\,x_3^2+(4\,x_1-8\,x_2)\,x_3+2\,x_2^2+4\,x_1\,x_2-4\,x_1^2)\,y_1^2+(-2\,x_2^2+4\,x_1\,x_2-2\,x_1^2)\,x_3^2+(4\,x_1\,x_2^2-8\,x_1^2\,x_2+4\,x_1^3)\,x_3-2\,x_1^2\,x_2^2+4\,x_1^3\,x_2-2\,x_1^4\right)y_2+ \\
& (-2\,x_3^2+4\,x_2\,x_3-2\,x_2^2)\,y_1^3+((-2\,x_2^2+4\,x_1\,x_2-2\,x_1^2)\,x_3^2+(4\,x_2^3-8\,x_1\,x_2^2+4\,x_1^2\,x_2)\,x_3-2\,x_2^4+4\,x_1\,x_2^3-2\,x_1^2\,x_2^2)\,y_1)\,y_3+(y_1^2+x_3^2-2\,x_1\,x_3+x_1^2)\,y_2^4+ \\
& ((-2\,x_3^2+4\,x_1\,x_3-2\,x_1^2)\,y_1-2\,y_1^3)\,y_2^3+ \\
& \left(y_1^4+(2\,x_3^2+(-2\,x_2-2\,x_1)\,x_3+2\,x_2^2-2\,x_1\,x_2+2\,x_1^2)\,y_1^2+x_3^4+(-2\,x_2-2\,x_1)\,x_3^3+(2\,x_2^2+2\,x_1\,x_2+2\,x_1^2)\,x_3^2+(-4\,x_1\,x_2^2+2\,x_1^2\,x_2-2\,x_1^3)\,x_3+2\,x_1^2\,x_2^2-2\,x_1^3\,x_2+x_1^4\right) \\
& y_2^2+\left((-2\,x_3^2+4\,x_2\,x_3-2\,x_2^2)\,y_1^3+(-2\,x_3^4+(4\,x_2+4\,x_1)\,x_3^3+(-2\,x_2^2-8\,x_1\,x_2-2\,x_1^2)\,x_3^2+(4\,x_1\,x_2^2+4\,x_1^2\,x_2)\,x_3-2\,x_1^2\,x_2^2)\,y_1\right)y_2+(x_3^2-2\,x_2\,x_3+x_2^2)\,y_1^4+ \\
& (x_3^4+(-2\,x_2-2\,x_1)\,x_3^3+(2\,x_2^2+2\,x_1\,x_2+2\,x_1^2)\,x_3^2+(-2\,x_2^3+2\,x_1\,x_2^2-4\,x_1^2\,x_2)\,x_3+x_2^4-2\,x_1\,x_2^3+2\,x_1^2\,x_2^2)\,y_1^2+(x_2^2-2\,x_1\,x_2+x_1^2)\,x_3^4+ \\
& (-2\,x_2^3+2\,x_1\,x_2^2+2\,x_1^2\,x_2-2\,x_1^3)\,x_3^3+(x_2^4+2\,x_1\,x_2^3-6\,x_1^2\,x_2^2+2\,x_1^3\,x_2+x_1^4)\,x_3^2+(-2\,x_1\,x_2^4+2\,x_1^2\,x_2^3+2\,x_1^3\,x_2^2-2\,x_1^4\,x_2)\,x_3+x_1^2\,x_2^4-2\,x_1^3\,x_2^3+x_1^4\,x_2^2)/ \\
& ((2\,x_2-2\,x_1)\,y_3+(2\,x_1-2\,x_3)\,y_2+(2\,x_3-2\,x_2)\,y_1)\,]
\end{aligned}
$$

Figure 3: Equation of a circle passing through three given points.

```
class Reflection
{
  double a0, b0, r0;

  public static double sqr(double x) {return x*x;}

// Reflect a point
  Point2D reflect(Point2D p)
  {
    double a=p.x; double b=p.y; double xx, yy;
    xx=        ((a-a0)*sqr(r0))/(sqr(b0-b)+sqr(a-a0))+a0;
    yy=        ((b-b0)*sqr(r0))/(sqr(b0-b)+sqr(a-a0))+b0;

    return new Point2D(xx, yy);
  }

  Reflection(double a, double b, double r)
  {
    a0=a; b0=b;r0=r;
  }

// Given three points on a Poincare geodesic (usually p3 is interior point on p1 p) calculates the coefficients of
     the circle inversion
// which gives the hyperbolic reflection
  Reflection(Point2D p1, Point2D p2, Point2D p3)
  {
    double a1, b1, a2, b2, a3, b3;
    a1=p1.x;
    b1=p1.y;
    a2=p2.x;
    b2=p2.y;
    a3=p3.x;
    b3=p3.y;

    a0 = -((b2-b1)*Math.pow(b3, 2)+((-Math.pow(b2, 2))+Math.pow(b1, 2)-Math.pow(a2, 2)+Math.pow(a1, 2))*b3+b1*Math.
         pow(b2, 2)+((-Math.pow(b1, 2))+Math.pow(a3, 2)-Math.pow(a1, 2))*b2+(Math.pow(a2, 2)-Math.pow(a3, 2))*b1)
         /((2*a2-2*a1)*b3
      +(2*a1-2*a3)*b2+(2*a3-2*a2)*b1);

    b0 = ((a2-a1)*Math.pow(b3, 2)+(a1-a3)*Math.pow(b2, 2)+(a3-a2)*Math.pow(b1, 2)+(a2-a1)*Math.pow(a3, 2)+(Math.pow
         (a1, 2)
      -Math.pow(a2, 2))*a3+a1*Math.pow(a2, 2)-Math.pow(a1, 2)*a2)/((2*a2-2*a1)*b3+(2*a1-2*a3)*b2+(2*a3-2*a2)*b1);

    r0 = Math.sqrt((Math.pow(b2, 2)-2*b1*b2+Math.pow(b1, 2)+Math.pow(a2, 2)-2*a1*a2+Math.pow(a1, 2))*Math.pow(b3,
         4)+
      ((-2*Math.pow(b2, 3))+2*b1*Math.pow(b2, 2)+(2*Math.pow(b1, 2)-2*Math.pow(a2, 2)+4*a1*a2-2*Math.pow(a1, 2))*b2
      -2*Math.pow(b1, 3)+((-2*Math.pow(a2, 2))+4*a1*a2-2*Math.pow(a1, 2))*b1)*Math.pow(b3, 3)+(Math.pow(b2, 4)+2*b1
           *Math.pow(b2, 3)
      +((-6*Math.pow(b1, 2))+2*Math.pow(a3, 2)+((-2*a2)-2*a1)*a3+2*Math.pow(a2, 2)-2*a1*a2+2*Math.pow(a1, 2))*Math.
           pow(b2, 2)
      +(2*Math.pow(b1, 3)+((-4*Math.pow(a3, 2))+(4*a2+4*a1)*a3+2*Math.pow(a2, 2)-8*a1*a2+2*Math.pow(a1, 2))*b1)*b2+
           Math.pow(b1, 4)
      +(2*Math.pow(a3, 2)+((-2*a2)-2*a1)*a3+2*Math.pow(a2, 2)-2*a1*a2+2*Math.pow(a1, 2))*Math.pow(b1, 2)+(2*Math.
           pow(a2, 2)-4*a1*a2+2*Math.pow(a1, 2))*Math.pow(a3, 2)
      +((-2*Math.pow(a2, 3))+2*a1*Math.pow(a2, 2)+2*Math.pow(a1, 2)*a2-2*Math.pow(a1, 3))*a3
      +Math.pow(a2, 4)-2*a1*Math.pow(a2, 3)+2*Math.pow(a1, 2)*Math.pow(a2, 2)-2*Math.pow(a1, 3)*a2+Math.pow(a1, 4))
           *Math.pow(b3, 2)+
      ((-2*b1*Math.pow(b2, 4))+(2*Math.pow(b1, 2)-2*Math.pow(a3, 2)+4*a1*a3-2*Math.pow(a1, 2))*Math.pow(b2, 3)+(2*
           Math.pow(b1, 3)
      +(2*Math.pow(a3, 2)+(4*a2-8*a1)*a3-4*Math.pow(a2, 2)+4*a1*a2+2*Math.pow(a1, 2))*b1)*Math.pow(b2, 2)
      +((-2*Math.pow(b1, 4))+(2*Math.pow(a3, 2)+(4*a1-8*a2)*a3+2*Math.pow(a2, 2)+4*a1*a2-4*Math.pow(a1, 2))*Math.
           pow(b1, 2)+((-2*Math.pow(a2, 2))+4*a1*a2-2*Math.pow(a1, 2))*Math.pow(a3, 2)
      +(4*a1*Math.pow(a2, 2)-8*Math.pow(a1, 2)*a2+4*Math.pow(a1, 3))*a3-2*Math.pow(a1, 2)*Math.pow(a2, 2)+4*Math.
           pow(a1, 3)*a2-2*Math.pow(a1, 4))*b2
      +((-2*Math.pow(a3, 2))+4*a2*a3-2*Math.pow(a2, 2))*Math.pow(b1, 3)+(((-2*Math.pow(a2, 2))+4*a1*a2-2*Math.pow(
           a1, 2))*Math.pow(a3, 2)+(4*Math.pow(a2, 3)-8*a1*Math.pow(a2, 2)+4*Math.pow(a1, 2)*a2)*a3-2*Math.pow(a2,
```

```
        4)
    +4*a1*Math.pow(a2, 3)-2*Math.pow(a1, 2)*Math.pow(a2, 2))*b1)*b3+(Math.pow(b1, 2)+Math.pow(a3, 2)-2*a1*a3+Math
        .pow(a1, 2))*Math.pow(b2, 4)+
    (((-2*Math.pow(a3, 2))+4*a1*a3-2*Math.pow(a1, 2))*b1-2*Math.pow(b1, 3))*Math.pow(b2, 3)+(Math.pow(b1, 4)+(2*
        Math.pow(a3, 2)
    +((-2*a2)-2*a1)*a3+2*Math.pow(a2, 2)-2*a1*a2+2*Math.pow(a1, 2))*Math.pow(b1, 2)+Math.pow(a3, 4)+((-2*a2)-2*a1
        )*Math.pow(a3, 3)
    +(2*Math.pow(a2, 2)+2*a1*a2+2*Math.pow(a1, 2))*Math.pow(a3, 2)+((-4*a1*Math.pow(a2, 2))+2*Math.pow(a1, 2)*a2
        -2*Math.pow(a1, 3))*a3+2*Math.pow(a1, 2)*Math.pow(a2, 2)-2*Math.pow(a1, 3)*a2+Math.pow(a1, 4))*Math.pow(
        b2, 2)
    +(((-2*Math.pow(a3, 2))+4*a2*a3-2*Math.pow(a2, 2))*Math.pow(b1, 3)+((-2*Math.pow(a3, 4))+(4*a2+4*a1)*Math.pow
        (a3, 3)+((-2*Math.pow(a2, 2))-8*a1*a2-2*Math.pow(a1, 2))*Math.pow(a3, 2)
    +(4*a1*Math.pow(a2, 2)+4*Math.pow(a1, 2)*a2)*a3-2*Math.pow(a1, 2)*Math.pow(a2, 2))*b1)*b2+(Math.pow(a3, 2)-2*
        a2*a3+Math.pow(a2, 2))*Math.pow(b1, 4)
    +(Math.pow(a3, 4)+((-2*a2)-2*a1)*Math.pow(a3, 3)+(2*Math.pow(a2, 2)+2*a1*a2+2*Math.pow(a1, 2))*Math.pow(a3,
        2)+((-2*Math.pow(a2, 3))
    +2*a1*Math.pow(a2, 2)-4*Math.pow(a1, 2)*a2)*a3+Math.pow(a2, 4)-2*a1*Math.pow(a2, 3)+2*Math.pow(a1, 2)*Math.
        pow(a2, 2))*Math.pow(b1, 2)+(Math.pow(a2, 2)-2*a1*a2+Math.pow(a1, 2))*Math.pow(a3, 4)
    +((-2*Math.pow(a2, 3))+2*a1*Math.pow(a2, 2)+2*Math.pow(a1, 2)*a2-2*Math.pow(a1, 3))*Math.pow(a3, 3)
    +(Math.pow(a2, 4)+2*a1*Math.pow(a2, 3)-6*Math.pow(a1, 2)*Math.pow(a2, 2)+2*Math.pow(a1, 3)*a2+Math.pow(a1, 4)
        )*Math.pow(a3, 2)
    +((-2*a1*Math.pow(a2, 4))+2*Math.pow(a1, 2)*Math.pow(a2, 3)+2*Math.pow(a1, 3)*Math.pow(a2, 2)-2*Math.pow(a1,
        4)*a2)*a3+Math.pow(a1, 2)*Math.pow(a2, 4)
    -2*Math.pow(a1, 3)*Math.pow(a2, 3)+Math.pow(a1, 4)*Math.pow(a2, 2))/((2*a2-2*a1)*b3+(2*a1-2*a3)*b2+(2*a3-2*a2
        )*b1);
  }
}
```

## References


[1] Michael P Hitchman. *Geometry with an introduction to cosmic topology*. Jones & Bartlett Learning, 2009.

[2] Frank Nielsen. *Visual computing: Geometry, graphics, and vision*. Charles River Media, Inc., 2005.

[3] Frank Nielsen and Richard Nock. Hyperbolic Voronoi diagrams made easy. In *2010 International Conference on Computational Science and Its Applications*, pages 74–80. IEEE, 2010.

[4] Frank Nielsen and Richard Nock. The hyperbolic Voronoi diagram in arbitrary dimension. *arXiv preprint arXiv:1210.8234*, 2012.

[5] Frederic Sala, Chris De Sa, Albert Gu, and Christopher Ré. Representation tradeoffs for hyperbolic embeddings. In *International conference on machine learning*, pages 4460–4469. PMLR, 2018.

[6] Toshihiro Tanuma, Hiroshi Imai, and Sonoko Moriyama. Revisiting hyperbolic Voronoi diagrams from theoretical, applied and generalized viewpoints. In *2010 International Symposium on Voronoi Diagrams in Science and Engineering*, pages 23–32. IEEE, 2010.