Structures cannot be avoided!
— Ramsey theory on the intersection graphs of line segments —

Frank Nielsen

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Consider n line segments $S_1, \ldots, S_n$ on the plane in *general position* (ie., segments are either disjoint or intersect in exactly one point). We ask the following question:
How many **pairwise disjoint segments** <u>OR</u> how many **pairwise intersecting segments** are there?

Define the *intersection graph* $G = (V, E)$ where each segment S_i is associated to a corresponding node V_i, and where there is an edge $\{V_i, V_j\}$ if and only if the corresponding line segments intersect ($S_i \cap S_j \neq \emptyset$). A subset of pairwise intersecting segments corresponds to a clique in G, and a subset of pairwise disjoint segments corresponds to an independent set in G (an anti-clique also called a stable). Consider the complement graph $\bar{G} = (V, \bar{E})$ where $\bar{E} = \mathcal{E} \backslash E$, with $\mathcal{E} = \{\{V_i, V_j\} \ : \ i \neq j\}$ the full edge set. $\bar{G}$ is the *disjointness graph* [3] of the segments (i.e., an edge between nodes if and only if corresponding segments are disjoint), and we have $G \cup \bar{G} = K_n$, the clique of size n.

Ramsey-type theorems are characterizing the following types of questions: "How large a structure must be to guarantee a given property?" Surprisingly, complete disorder is impossible! That is, there always exists (some) order in structures!

Define the **Ramsey number** $R(s,t)$ as the minimum number n such that any graph with $|V| = n$ nodes contains either an independent set of size s or a clique K_t of size t.

One can prove that those Ramsey numbers are all *finite* [4] (by proving the recursive formula $R(s,t) \leq R(s-1,t) + R(s,t-1)$ with terminal cases $R(s,1) = R(1,t) = 1$ for $s,t \geq 1$), and that the following bound holds (due to the theorem of Erdös-Szekeres [2]):

$$R(s,t) \leq \binom{s+t-2}{s-1} < \infty.$$

When $s = t$, $\binom{s+t-2}{s-1} = \binom{2(s-1)}{s-1}$ is a central binomial coefficient that is upper bounded by 2^{2s}. Therefore the diagonal Ramsey number $R(s) = R(s,s)$ is upper bounded by 2^{2s}. Furthermore, we have the following lower bound: $2^{\frac{s}{2}} < R(s)$ when $s \geq 3$ [2]. Thus $s \geq \lfloor \frac{1}{2} \log_2 n \rfloor$. Erdös proved using a probabilistic argument [1] that there exists a graph G such that $s \leq 2 \log_2 n$ (G and $\bar{G}$ do not contain K_s subgraphs). It is proved in [3] (2017) a much stronger result that $s = \Omega(n^{\frac{1}{5}})$ for intersection graphs of line segments: Thus there are always $\Omega(n^{\frac{1}{5}})$ pairwise disjoint or pairwise intersecting segments in a set of n segments in general position.

In general, one can consider a **coloring** of the edges of the clique K_n into c colors, and asks for the largest **monochromatic clique** K_m in the edge-colored K_n. For the pairwise disjoint/intersecting line segments, we have $c = 2$: Say, we color edges red when their corresponding segments intersect and blue, otherwise.

Ramsey's theorem [4] (1930) states that for all c, there exists $n \geq m \geq 2$ such that every c-coloring of K_n has a monochromatic clique K_m.

Let us conclude with the theorem on acquaintances (people who already met) and strangers (people who meet for the first time): In a group of six people, either at least three of them are pairwise mutual strangers or at least three of them are pairwise mutual acquaintances. Consider K_6 ($n = 6$, 15 edges), and color an edge in red if the edge people already met and in blue, otherwise. Then there is a monochromatic triangle ($m = 3$). Proof: $R(3) = R(3,3) \leq \binom{4}{2} = 6$.

Well, it is known that $R(4) = 18$ but $R(5)$ is not known! We only know that $43 \leq R(5) \leq 48$, $102 \leq R(6,6) \leq 165$, etc. Quantum computers [5] can be used to compute Ramsey numbers!

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References

[1] Paul Erdös. Some remarks on the theory of graphs. *Bulletin of the American Mathematical Society*, 53(4):292–294, 1947.

[2] Paul Erdös and George Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[3] J. Pach, G. Tardos, and G. Toth. Disjointness graphs of segments. *ArXiv e-prints*, April 2017.

[4] Frank P Ramsey. On a problem of formal logic. In *Classic Papers in Combinatorics*, pages 1–24. Springer, 2009.

[5] Hefeng Wang. Determining Ramsey numbers on a quantum computer. *Physical Review A*, 93(3), 2016.