Thales circle theorem extended to Mahalanobis geometry

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1 Thales' theorem in Euclidean geometry

In planar Euclidean geometry, Thales' theorem states that any triangle pqr circumscribing a circle with one pair (p,q) of antipodal points is necessarily a right triangle. A pair (p,q) of antipodal points of a smooth convex object is such that the tangent lines at p and q are parallel to each other. See Figure 1 for an illustration, and [3] for a historical account (Thales of Miletus, 624–546 BC).

Theorem 1 (Thales' circle theorem) *Any triangle circumscribed by a circle with one side being a diameter is right-angle.*

2 Thales' theorem in Mahanalobis geometry

Let $D_A(p,q)$ denote the Mahalanobis distance between two points p and q, for a positive definite matrix $A \succ 0$:

$$D_A(p,q) = \sqrt{(p-q)^\top A (p-q)} = \|p-q\|_A.$$

When $A = I$ is the 2×2 identity matrix, the Mahalanobis distance amounts to the Euclidean distance $D_E(p,q) = \|p-q\| = \sqrt{(p-q)^\top (p-q)}$.

A Mahalanobis circle [2] $C_A(c,r)$ of center c and radius r is defined as follows:

$$C_A(c,r) = \{x \ : \ D_A(c,x) = r\}.$$

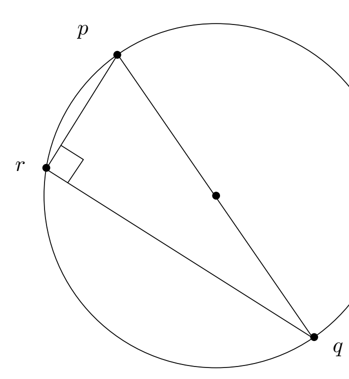

Figure 1: Thales' circle theorem: Triangle pqr is right-angle at r where $[pq]$ is a diameter. (p,q) is a pair of antipodal points.

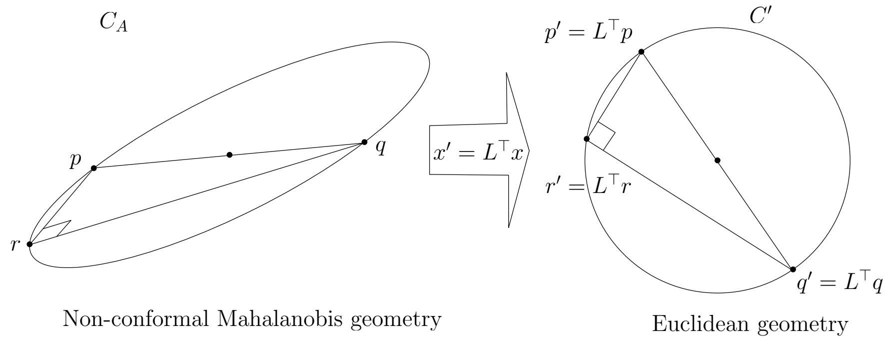

Figure 2: Thales' circle theorem in Mahalanobis geometry: Triangle pqr is right-angle at r where $[pq]$ is an antipodal pair of points. Notice that Mahalanobis geometry is generally not conformal so that a Mahalanobis right-angle does not visualize as a Euclidean right-angle.

A Mahalanobis circle has an ellipsoid (Euclidean) shape.

Let us generalize Thales' theorem as follows:

Theorem 2 (Thales' Mahalanobis circle theorem) *Any triangle circumscribed by a Mahalanobis circle with one pair of points being antipodal is right-angle.*

Proof: Consider the Cholesky decomposition of A: $A = LL^\top = U^\top U$ with L ($U = L^\top$) a lower triangular matrix (an upper triangular matrix, respectively) with positive diagonal elements. The Mahalanobis distance amounts to calculate an ordinary Euclidean distance on affinely transformed points $x' = L^\top x = Ux$:

$$\begin{aligned} D_A(p,q) &= \sqrt{(p-q)^\top LL^\top (p-q)}, \\ &= D_E(L^\top p, L^\top q). \end{aligned}$$

Thus a Mahalanobis circles C_A transforms affinely to a Euclidean circle $C_E = C_I$, and antipodal pairs of points on C_A remain antipodal in C_E.

Two vectors u and v are perpendicular in the Mahalanobis geometry if and only if $u^\top Av = 0$. That is, if $u^\top Av = u^\top LL^\top v = (L^\top u)^\top L^\top v = {u'}^\top v' = 0$.

A triangle pqr circumscribing the Mahalanobis circle C_A with (p,q) an antipodal pair in Mahalanobis geometry transforms into a triangle $p'q'r'$ circumscribing the Euclidean circle $C' = \{L^\top x \ : \ x \in C_A(c,r)\}$ with (p',q') an antipodal pair. Therefore $p'q'r'$ is a right-angle triangle in Euclidean geometry, and:

$$\begin{aligned} (q'-r')^\top (p'-r') &= 0, & (1) \\ (L^\top (q-r))^\top L^\top (p-r) &= 0, & (2) \\ (q-r)^\top LL^\top (p-r) &= 0, & (3) \\ (q-r)A(p-r) &= 0. & (4) \end{aligned}$$

Therefore, pqr is a right-angle triangle in Mahalanobis geometry. □

Note that Mahalanobis geometry is not conformal when $A \neq \lambda I$ (for $\lambda > 0$), the scaled identity matrix. Therefore angles are not preserved in Mahalanobis geometry: That is, a Mahalanobis right-angle cannot be visualized as a Euclidean right-angle in general.

Squared Mahalanobis distances are the only symmetric Bregman divergences [1]. But Thales' theorem do not extend to other (asymmetric) Bregman divergences.

References

[1] Jean-Daniel Boissonnat, Frank Nielsen, and Richard Nock. Bregman Voronoi diagrams. *Discrete & Computational Geometry*, 44(2):281–307, 2010.

[2] Frank Nielsen and Richard Nock. On the smallest enclosing information disk. *Information Processing Letters*, 105(3):93–97, 2008.

[3] Christoph J Scriba and Peter Schreiber. Geometry in the Greek-Hellenistic era and late antiquity. In *5000 Years of Geometry*, pages 27–116. Springer, 2015.