The harmonic mean of two independent Cauchy distributions is a Cauchy distribution

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December 2021

Abstract

In this note, we prove the well-known fact [1] that the Harmonic mean of two Cauchy distributions is a Cauchy distribution.

Consider $C_1 \sim \mathrm{Cauchy}(l_1, s_1)$ and $C_2 \sim \mathrm{Cauchy}(l_2, s_2)$ two independent Cauchy distributions. Then their harmonic mean defined by

$$C_{12} = \frac{1}{\frac{1}{2}\frac{1}{C_1} + \frac{1}{2}\frac{1}{C_2}} = \frac{2\,C_1 C_2}{C_1 + C_2}$$

follows a Cauchy distribution.

The proof is based on the following three properties of Cauchy distributions:

- Let $C \sim \mathrm{Cauchy}(l, s)$ then $\frac{1}{C} \sim \mathrm{Cauchy}\left(\frac{l}{l^2+s^2}, \frac{s}{l^2+s^2}\right) = \frac{1}{l^2+s^2}\mathrm{Cauchy}(l, s)$.
- Let $C \sim \mathrm{Cauchy}(l, s)$ then $\lambda C \sim \mathrm{Cauchy}(\lambda l, \lambda s)$.
- Let $C_1 \sim \mathrm{Cauchy}(l_1, s_1)$ and $C_2 \sim \mathrm{Cauchy}(l_2, s_2)$ be two independent Cauchy distributions. Then $C_1 + C_2 \sim \mathrm{Cauchy}(l_1 + l_2, s_1 + s_2)$.

It follows that $C_{12} \sim \mathrm{Cauchy}(l_{12}, s_{12})$ with

$$l_{12} = 2\frac{(l_1 s_2^2 + l_2 s_1^2 + l_1 l_2^2 + l_1^2 l_2)}{(l_1 + l_2)^2 + (s_1 + s_2)^2}, \quad s_{12} = 2\frac{(s_1 s_2^2 + (s_1^2 + l_1^2)s_2 + l_2^2 s_1)}{(l_1 + l_2)^2 + (s_1 + s_2)^2}.$$

The following code below in R illustrates the result:

```
# install.packages("univariateML")
library("univariateML")
n <- 100000
l1 <- 1.5
s1 <- 1
l2 <- 2
s2 <- 3
x1 <- rcauchy(n,l1,s1)
x2 <- rcauchy(n,l2,s2)
h12<- 2*x1*x2/(x1+x2)
mlcauchy(h12)
```

```
#l12
2*(l1*s2*s2+l2*s1*s1+l1*l2*l2+l1*l1*l2)/((s1+s2)*(s1+s2)+(l1+l2)*(l1+l2))
#s12
2*(s1*s2*s2+(s1*s1+l1*l1)*s2+l2*l2*s1)/((s1+s2)*(s1+s2)+(l1+l2)*(l1+l2))
```

More generally, Menon [1] (§3) proved that for n independent Cauchy distributions $C_i \sim \mathrm{Cauchy}(l_i, s_i)$, we have:

$$H = \frac{1}{\sum_{i=1}^{n} \frac{1}{C_i}} \sim \mathrm{Cauchy}\left(\frac{l}{l^2+s^2}, \frac{s}{l^2+s^2}\right),$$

with

$$\begin{aligned} l &= \sum_{i=1}^{n} \frac{l_i}{l_i^2+s_i^2}, \\ s &= \sum_{i=1}^{n} \frac{s_i}{l_i^2+s_i^2}. \end{aligned}$$

Thus the harmonic mean of n independent Cauchy distributions C_i's is a Cauchy distribution $\mathrm{Cauchy}\left(\frac{nl}{l^2+s^2}, \frac{ns}{l^2+s^2}\right)$.

Acknowledgments. The author is indebted to Professor Kazuki Okamura (Shizuoka University, Japan) who pointed out reference [1].

References

[1] MV Menon. A characterization of the Cauchy distribution. *The Annals of Mathematical Statistics*, 33(4):1267–1271, 1962.