Building an Apollonian circle packing

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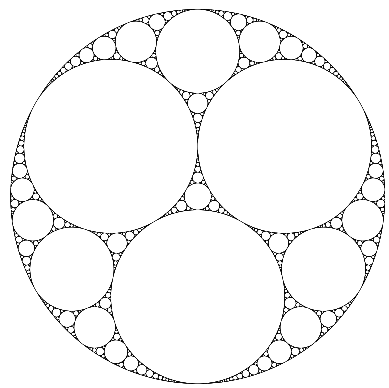

Figure 1: The Apollonius gasket: A fractal geometric object of Haussdorf dimension [5] $\simeq 1.3057$.

1 Introduction

Apollonius of Perga (262-190 BC) devoted a book on conics [3] which considered geometric construction problems with mutually tangent circles. The so-called *Apollonius gasket* depicted in Figure 1 is a fractal which is built by considering recursively the inner tangent circle to three kissing circles (mutually touching circles). In this note, we show how to build (and program) such a gasket or Apollonian circle packing [1].

2 Building three kissing circles

To build three kissing circles C_1, C_2, and C_3 (Figure 2), we proceed as follows:

- Choose at random two distinct points p_1 and p_2 for the circle centers of C_1 and C_2, respectively,
- Calculate their Euclidean distance $d_{12} = \|p_1 - p_2\|$,
- choose at random a proportion $\alpha \in (0, 1)$, and let the radii of circles C_1 and C_2 be $r_1 = \alpha d_{12}$ and $r_2 = (1 - \alpha)d_{12}$, respectively (thus C_1 is kissing C_2).
- choose at random a radius $r_3 > 0$, and calculate the two intersection points i_1 and i_2 of the two intersecting circles $C_1' = \mathrm{circle}(p_1, r_1 + r_3)$ and $C_2' = \mathrm{circle}(p_2, r_2 + r_3)$. Let $C_3 = \mathrm{circle}(i_1, r_3)$. The three circles C_1, C_2 and C_3 are mutually touching.

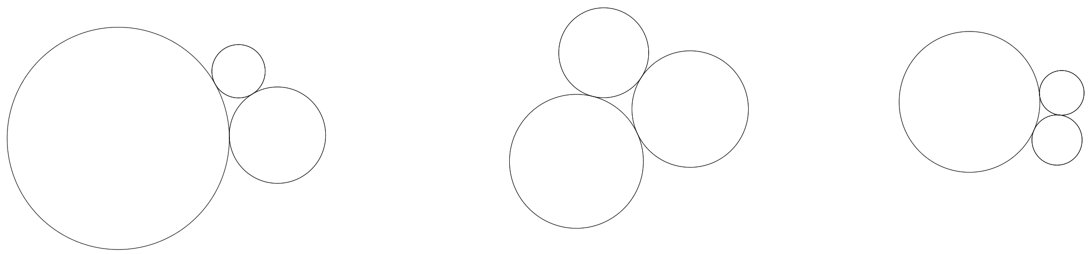

Figure 2: Examples of mutually kissing circles

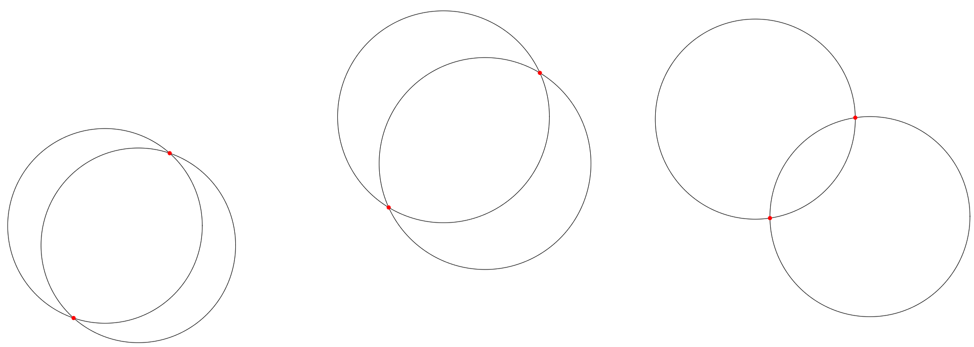

Figure 3: Examples of pairs of intersecting circles with their two intersection points.

The two intersection points i_1 and i_2 of two intersecting circles $C_1 = \mathrm{circle}(p_1, r_1)$ and $C_2 = \mathrm{circle}(p_2, r_2)$ (with respective centers $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$) are calculated as follows: We consider the radical axis (difference of the two circle equations yielding a line equation), and calculate the intersection of the radical axis with C_1 by solving a quadratic equation. Overall, we find the following intersection points:

$$\begin{aligned}
i_1 &= \bar{p} + \frac{r_1^2 - r_2^2}{2d_{12}^2} p_{21} + \frac{q_{21}}{2}\sqrt{\frac{2}{d_{12}^2}(r_1^2 + r_2^2) - \frac{(r_1^2 - r_2^2)^2}{d_{12}^4} - 1}, \\
i_2 &= \bar{p} + \frac{r_1^2 - r_2^2}{2d_{12}^2} p_{21} - \frac{q_{21}}{2}\sqrt{\frac{2}{d_{12}^2}(r_1^2 + r_2^2) - \frac{(r_1^2 - r_2^2)^2}{d_{12}^4} - 1},
\end{aligned}$$

where

$$\begin{aligned}
\bar{p} &= \frac{p_1 + p_2}{2}, \\
p_{21} &= p_2 - p_1, \\
q_{21} &= (y_2 - y_1, x_2 - x_1).
\end{aligned}$$

Figure 2 displays some examples of pairwise kissing circles. Figure 3 displays the two intersection points of intersecting circles.

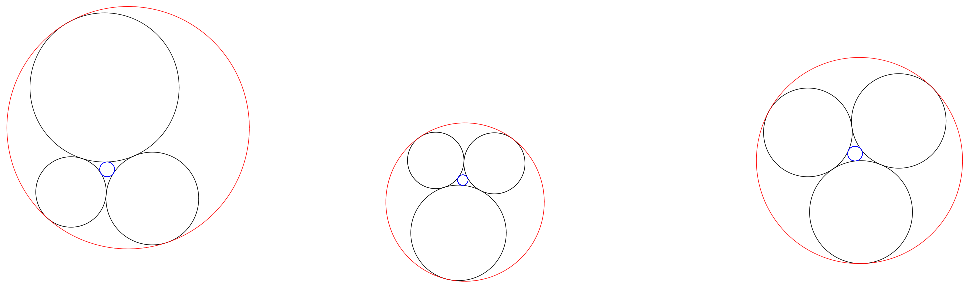

Figure 4: Examples of Soddy inner/outer circles to three kissing circles.

3 Descartes' theorem and complex Descartes' theorem

Given three kissing circles C_1, C_2, and C_3, we can find a fourth kissing circle C_4 as follows: Let $\kappa_1 = \frac{1}{r_1}$, $\kappa_2 = \frac{1}{r_2}$, and $\kappa_3 = \frac{1}{r_3}$ denote the curvatures of the three kissing circles C_1, C_2, and C_3. The curvatures of circles are positive. Then the signed curvature of a fourth kissing circle C_4 is given by Descartes' theorem (1643):

$$\kappa_4 = \kappa_1 + \kappa_2 + \kappa_3 + 2\sqrt{\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1}, \tag{1}$$

$$\kappa_5 = \kappa_1 + \kappa_2 + \kappa_3 - 2\sqrt{\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1}. \tag{2}$$

We have $\kappa_4 > 0$, and $r_4 = \frac{1}{\kappa_4}$ is the radius of the fourth kissing *inner* circle contained in the curved triangle defined by the three kissing circles. We have $r_5 = \frac{1}{|\kappa_5|}$, and r_5 is the radius of the *outer* fourth kissing circle the three kissing circles (circumscribing or not the circles C_1, C_2 and C_4).

Descartes' theorem has been rediscovered several times and published, including in a form of a *poem* entitled "The kiss precise" by Soddy (published in the celebrated Nature journal in 1936 [7]).

Once the inner and outer tangential circle radii have been computed, we can retrieve the center p_4 of $C_4 = \mathrm{circle}(p_4, r_4)$ and the center p_5 of $C_5 = \mathrm{circle}(p_5, r_5)$ using the *complex Descartes' theorem* [6]: First, we convert the Cartesian coordinates of the centers p_i to their equivalent complex numbers $z_i = x_i + iy_i$, for $i \in \{1, 2, 3, 4, 5\}$. Then we consider the two centers:

$$a = \frac{z_1\kappa_1 + z_2\kappa_2 + z_3\kappa_3 + 2\sqrt{\kappa_1\kappa_2 z_1 z_2 + \kappa_2\kappa_3 z_2 z_3 + \kappa_1\kappa_3 z_1 z_3}}{\kappa_4}, \tag{3}$$

$$b = \frac{z_1\kappa_1 + z_2\kappa_2 + z_3\kappa_3 - 2\sqrt{\kappa_1\kappa_2 z_1 z_2 + \kappa_2\kappa_3 z_2 z_3 + \kappa_1\kappa_3 z_1 z_3}}{\kappa_4} \tag{4}$$

where $\sqrt{z} = \sqrt{|z|}e^{i\frac{\theta(z)}{2}}$ (complex square root), $|z| = \sqrt{x^2 + y^2}$ (modulus) and $\theta(z) = \arctan\left(\frac{y}{x}\right)$ (phase).

If $\|a - z_1\| = r_1 + \frac{1}{\kappa_4}$ then $z_4 = a$ and $z_5 = b$. Otherwise $z_4 = b$ and $z_5 = a$.

Figure 4 displays some examples of tangential inner and outer circles.

4 Drawing the Apollonius gasket

Let C_1, C_2, C_3 be three kissing circles. We build the fractal by calling $\mathrm{Gasket}(C_1, C_2, C_3, l)$ where l denotes the level of recursion (e.g., $l = 5$). The procedure $\mathrm{Gasket}(C_1, C_2, C_3, l)$ stops when $l = 0$. Otherwirse the procedure builds the inner tangential circle C_4, draws it, and calls recursively $\mathrm{Gasket}(C_4, C_1, C_2, l-1)$, $\mathrm{Gasket}(C_4, C_1, C_3, l-1)$ and $\mathrm{Gasket}(C_4, C_2, C_3, l-1)$. Figure 5 displays some examples of Apollonius gaskets for $l = 4$ inner to kissing triangles.

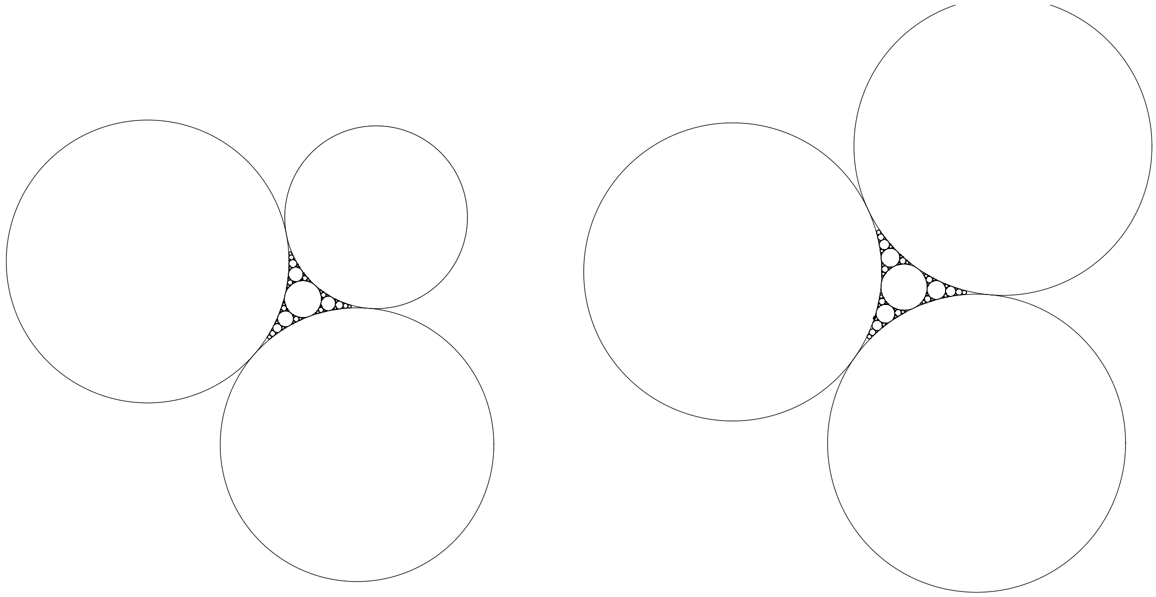

Figure 5: Examples of inner Apollonius gaskets.

Descartes' theorem has been generalized to arbirary dimensions [4] and various generalizations of the Apollonian circle packings have been studied [2].

References

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