Computational information geometry on Bregman manifolds and submanifolds

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Abstract

In the first part, we review the construction of a Bregman manifold (M, g, ∇, ∇^*) from a Legendre-type convex function [2] $F : \Theta \to \mathbb{R}$. The Riemannian metric tensor can be expressed as $g(\theta) = \nabla^2 F(\theta)$ or equivalently in the dual coordinate system $\eta = \nabla F(\theta)$ induced by the Legendre transform $F^*(\eta) = \langle \theta, \eta \rangle - F(\theta)$ as $g(\eta) = \nabla^2 F^*(\eta)$. The torsion-free flat affine connections ∇ and ∇^* are dual with respect to the metric tensor g since $\frac{\nabla+\nabla^*}{2}$ coincide with the Levi-Civita connection induced by g. By further using a representation function $r(\theta)$, we show that α-divergences are representational Bregman divergences on the positive orthant cone and curved representational Bregman divergences on the probability simplex [4]. In the second part, we show that symmetrized Bregman divergences are curved Bregman divergences, and present a Bregman projection theorem for calculating the curved Bregman centroid [3]. In the third and last part, we describe clustering [5], nearest-neighbor query data structures [6], and Voronoi diagrams [1] on Bregman manifolds and sub-manifolds with several applications in statistics and data science. Finally, we present work in progress pyBregMan: A Python library for algorithms and data-structures on BREGman MANifolds[7].

References

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