Schrödinger's problem, HJB equations and regularized Mass Transportation

July 2020

Les Houches Summer Week on Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning

Jean-Claude Zambrini
Grupo de Física-Matemática Univ. Lisboa (GFMUL)
Dept. de Matemática, Universidade de Lisboa

FCT Fundação para a Ciência e a Tecnologia

1. Introduction

This talk is about a single, very general, idea between Statistical and Quantum Mechanics, due to E. Schrödinger (1931-32). We shall illustrate how it has been around, implicitly, for a long time although periodically misunderstood or forgotten. It is closely related with Feynman's path integral approach (1949) and with Kantorovich regularization of Monge problem (1940). Today it is at the origin of a randomized version of Mechanics, whose existence was first suggested by S. Bernstein (1932).

2. Feynman's transition amplitude (1948-49)

The shortest way is to start from a key result of Feynman's approach generally forgotten.

Consider a classical Lagrangian $L(\omega, \dot{\omega}) = \frac{1}{2}|\dot{\omega}|^2 - V(\omega)$, V bounded below, $\omega \in C^2([s,u], \mathbb{R}^3)$, one of consequences of Feynman integration by parts formula for his "Functional Calculus" is the time discretized, $\forall t$

$$(CR)\ \left\langle \omega_j(t)\Big(\frac{\omega(t)-\omega(t-\Delta t)}{\Delta t}\Big)_k \right\rangle_{S_L} - \left\langle \Big(\frac{\omega(t+\Delta t)-\omega(t)}{\Delta t}\Big)_k \omega_j(t) \right\rangle_{S_L}$$

$$Q^j \qquad P^k \qquad\qquad\qquad P^k \qquad Q^j$$

$$= i\hbar\delta_{jk}\ ,$$

$1 \le j, k \le 3$, where $t \to \omega(t)$, now, denotes un unspecified “diffusion process" associated with L; $\hbar$ is a deformation parameter, Planck’s constant.

S_L refers to the classical Action functional of the system to be quantized

$S_L[\omega(\cdot); u-s] = \int_s^u L(\omega(t), \dot{\omega}(t))dt$

(CR) is difference of “expectations" of two time discretized functionals F of the paths ω computed using Feynman’s general formula for any “transition element" between any quantum states $\Psi_s, \bar{\varphi}_u$ (complex conjugate of φ_u) :

$$\langle F\rangle_{S_L} = \int\int\int \psi_s(x)\exp\frac{i}{\hbar}S_L[\omega; u-s]F[\omega(\cdot)]\bar{\varphi}_u(z)\mathcal{D}\omega dxdz$$

where $\mathcal{D}\omega$ denotes informally $\Pi_{s<t<u}d\omega(t)$ on $\{\omega \in C[s,u]\ |\omega(s)=x, \omega(u)=z\}$.

For us the proper way to look at boundary states ψ_s and $\bar{\varphi}_u$ in L^2, is as initial condition of

$$i\hbar\,\partial_t\psi = H\,\psi$$

(respectively final one of $-i\hbar\,\partial_t\bar{\varphi} = H\bar{\varphi}$) for

$$H = -\frac{\hbar^2}{2}\Delta + V$$

When $F = 1$, and in term of H ,

$$\langle 1 \rangle_{S_L} = \int \int \psi_s(x) \underbrace{e^{-\frac{i}{\hbar}(u-s)H}(x,z)}_{k(s,x,u,z)} \bar{\varphi}_u(z) \, dxdz = \langle \varphi_u | \psi_u \rangle_2,$$

a L^2 scalar product, after integration in x.

This is a complex number, without probabilistic interpretation. Key afterwards, a

"Transition amplitude"

What (CR) tells us :" Typical quantum paths are continuous but non differentiable" i.e Brownian-like. Feynman suggests to substitute the non-commutative calculus of self-adjoint operators in L^2 , ($Q^j P^k - P^k Q^j$ in (CR)) by a commutative calculus of diffusions (Itô calculus !) i.e, geometrically, a section of Phase space, where Q is promoted to a random process.

A "stochastic variational calculus" is associated , s.t $\lim_{\hbar \downarrow 0} \simeq$ least Action principle for S_L.

But : we know since the sixties (R.H. Cameron) that $e^{\frac{i}{\hbar} S_L} \mathcal{D}\omega$ is not a well defined (complex) measure so $\lim_{\Delta t \to 0}$ (discretization) and $< \cdot >_{S_L}$ do not exist.

Still, Feynman's calculus is a useful computational tool, specially when smooth trajectories $q(\cdot)$ solving least Action principle are known. For ex. $V = 0$, i.e $H = H_0$, 1 d, on $[0, u]$:

$$< \omega(t_1)\omega(t_2) >_{S_L} = q(t_1)q(t_2) + i\hbar \underset{\downarrow}{\frac{t_1(u-t_2)}{u}} \qquad 0 < t_1 < t_2 < u \quad (\text{and } t_1 \to t_2 \text{ for } t_2 < t_1)$$

random deformation of classical expression $q(\cdot)$.
Deformation parameter $= \hbar$.

Here $q(\cdot)$ is the straight line between x and z (geodesic) for $\psi_s = \delta_x, \ \bar{\varphi}_u = \delta_z$

Traditional ("Euclidean") way out of the doldrums : $t \to -it$ and Feynman-Kac formula (1949) for associated heat equation.

This is not the Euclidean method advocated by Schrödinger 18 years before Kac.

2. Schrödinger's problem (inspired by A. Eddington) 1931-32

A system of 1 d Brownian particles $X_t = \hbar^{\frac{1}{2}} W_t$, for $W_.$ a Wiener , $\hbar > 0$, evolves on the line during a time interval $[s, u]$. If an initial probability density is given, say $\mu_s = \rho_s(x)dx$ the future one is $P(X_t \in dz) = \eta_t^*(z)dz$, for η_t^* positive solution of

$$-\hbar\, \partial_t \eta^* = H_0 \eta^*$$

with $H_0 = -\frac{\hbar^2}{2}\Delta$ and $\eta_s^*(x) = \rho_s(x)$.

Typical irreversible dynamics. But what if the observed final probability density is $\mu_u(dz) = \rho_u(z)dz \neq \eta_u^*(z)dz$?

Schrödinger asks : What is the most probable evolution
$\rho_t(q)dq, \;\; s \leq t \leq u$ between ρ_s, ρ_u ?

Using an entropy extremization of Statistical physics, his conclusion is that the optimal (Markovian) process should be s.t. (B a Borelian)

$$P(X_t \in B) = \int_B \eta_t^*(q)\eta_t(q)dq \qquad\qquad s \leq t \leq u$$

where η_t^* solves the eq. above for <u>some</u> $\eta_s^* > 0$ and η_t the solution of <u>adjoint</u> eq.

$$\hbar\, \partial_t \eta = H_0 \eta$$

for <u>some</u> $\eta_u > 0$. This was Schrödinger's motivation. Informally, when $t \to it$ this probability becomes a scalar product in L^2, like $< 1 >_{S_L}$ of Feynman. But, here, X_t will be well defined.

<u>NB :</u> Informally, under $t \to -t, \; \eta_t^* \to \eta_t$ so the optimal stochastic dynamics should be “time reversible".

Why optimal "Markovian" process ?

In 1931 diffusion processes in Kolmogorov sense were not yet familiar. Bernstein, first probabilist reader of Schrödinger, understood that those optimal processes should not, in general, be Markovian. He also guessed (1932) that they should be probabilistic counterparts of classical solutions of least Action principle, suggesting the existence of a "Random Mechanics" founded on them.

(What I call these days the program of "Stochastic Deformation" JCZ (2015), ($\hbar \equiv$ deformation parameter). A presentation for theoretical physicists is in K.L. Chung, JCZ (2003).

For this class of processes, Bernstein defined the property needed :

$\mathcal{P}_t =$ usual increasing σ-algebra (Past information) not sufficient, $t \in [s, u]$ (Cf transition amplitude).
Take as well $\mathcal{F}_t$ decreasing (Future), $s \leq t \leq u$.

Def. $A \in \mathcal{P}_s \cup \mathcal{F}_t$, $B \in \sigma(s, t)$ (σ-algebra on $[s, t]$) then

$$P(A.B|\mathcal{P}_s \cup \mathcal{F}_t) = P(A|X_s, X_t).P(B|X_s, X_t) \qquad \forall s \leq t$$

Bernstein "Reciprocal property" (1932 !) (1 d Markov field or local Markov of Quantum Field Theory $\sim$ 1970). For more cf Léonard, Roelly, JCZ (2014).

Usual Markovian measure construction: given any initial prob. density and (forward, $s < t$) transition probability $\longrightarrow$ Finite dim. probabilities then Kolmogorov limit of time discretization.

Replace (s,t) transition probability by 3 times $s \leq t \leq u$, Borelian $B \to Q(s,x,t,B,u,z)$ s.t for z fixed $Q_{uz}(s,x,t,B) =$ forward Markov transition, and for x fixed $Q_{sx}(t,B,u,z) =$ backward Markovian transition (NB: Time-symmetry of Markov property), Kolmogorov data of initial probability by a joint one $dM(x,z)$. Then

Theorem (1974, Jamison)

Given Q and M, $\exists$ Prob. measure P_M s.t under P_M, X_t is Bernstein,
$P_M(X_s \in B_s, X_u \in B_u) = M(B_s \times B_u)$ *and*
$P_M(X_s \in B_s, X_{t_1} \in B_1, ..., X_{t_n} \in B_n, X_u \in B_u) =$
$\int_{B_s \times B_u} dM(x,z) \int_{B_1} Q(s,x,t_1,dq_1,u,z) ... \int_{B_n} Q(t_{n-1},q_{n-1},t_n,dq_{n-1},u,z)$,
$s < t_1 < ... < t_n < u$

(one could also fix, instead, (s,x))

NB: Resulting measures generally non-Markovian. But pick Euclidean version of Feynman Markovian “Transition amplitude":

$$M_M(B_s \times B_u) = \int_{B_s \times B_u} \eta_s^*(x) \underbrace{e^{-\frac{1}{\hbar}(u-s)H}(x,z)}_{h(s,x,u,z)>0} \eta_u(z) dxdz$$

for $\eta_s^*, \eta_u > 0$, BC of the 2 adjoint heat Eqs. with $H = H_0 + V(q)$ with V bounded below in Kato's class.

In terms of underlying integral kernel $h(s,x,u,z) = e^{-\frac{1}{\hbar}(u-s)H}(x,z)$, M_M results from

$$Q(s,x,t,dq,u,z) = h^{-1}(s,x,u,z)h(s,x,t,q)h(t,q,u,z)dq$$

in Theorem.

How to describe variationally, as Bernstein and Feynman suggested, the dynamics of Bernstein processes $X.$ for H as before ?

Feynman's $S_L[\omega(\cdot); u-s] = \int_s^u L(\omega(t), \dot{\omega}(t))dt$ divergent along $t \to \dot{X}_t$.

But any such diffusion admits a $\mathcal{P}_t$ (Itô's) regularization of right-hand side derivative :

$$D_t X = \lim_{\Delta t \downarrow 0} E_t \Big[\frac{X_{t+\Delta t} - X_t}{\Delta t} \Big],$$

($E_t =$ conditional expectation given X_t : $E[\ ...\ |X_t]$) i.e usual forward drift.

NB: OK for Markovian Bernstein. Then $D_t X = B(X_t, t)$, for some B.

Duality $\Rightarrow$ defines as well $D_t^* X = \lim_{\Delta t \downarrow 0} E_t \big[\frac{X_t - X_{t-\Delta t}}{\Delta t} \big] = B^*(X_t, t)$ ($\mathcal{F}_t$-regularization)

More generally, $D_t f(X_t, t), D_t^* g(X_t, t)$ defined by Itô calculus $\forall f, g$. Drifts for $f = g = X_t$.

What about $L(\omega, \dot{\omega})$ for H as before ? Schrödinger's problem is Euclidean $\rightarrow L(X_t, DX_t) = \frac{1}{2}|DX_t|^2 + V(X_t) \; =$ simplest choice.

Define the regularized (with Bolza final condition S_u) Action functional

$$J[X(\cdot), u-t] = E_t\{\int_t^u (\frac{1}{2}|DX_\tau|^2 + V(X_\tau))d\tau + S_u(X_u)\}$$

on the domain

$$\mathcal{D}_J = \{X. << P_W^\hbar, \text{ with fixed diffusion coefficient } \hbar\,\mathbb{I}, \text{ unknown drift } B\}$$

Various stochastic variational principles are known, to find $X_{\cdot} \in \mathcal{D}_J$ minimizing J. Two of them :

1) Controlled Markov processes (Ref. W.H Fleming, H.M. Soner 2006)

Admissible Control $= D_\tau X \in C$, $L(X, DX) =$ "Running cost",
$S_u =$ "Terminal cost"

$S_L(x,t) = \inf_C J[x,t,\ \text{control}]$ where $E_{x,t} = E[...|X_t = x]$ is used now
$= $ "Value function"

Bellman's principle of Dynamic Programming (for L as before)
$\Rightarrow S_L(x,t)$ satisfies HJB Eq , i.e the stochastic deformation of Hamilton-Jacobi Eq.
(for our system) :

$$(HJB) \quad -\frac{\partial S_L}{\partial t} + \frac{1}{2}|\nabla S_L|^2 - \frac{\hbar}{2}\Delta S_L - V = 0 \qquad \text{with } S_L(q,u) = S_u(q)$$

and minimizing $D_\tau X = B(X,\tau) = -\nabla S_L(X,\tau),\ \tau \in [t,u]$

NB: $S_L(x,t) = -\hbar \log \eta(x,t)$

2) Relation with Schrödinger's problem and classical variational principles :

Def an extremal X of J by $E_{qt}\left[\lim_{\varepsilon\to 0}\frac{J[X+\varepsilon\delta X]-J[X]}{\varepsilon}\right]=\delta J[X](\delta X)=0$ $\forall\delta X\in$ Cameron-Martin Hilbert space. Then, $\forall\delta X$, by Itô calculus

$$E_{qt}\int_t^u\Big(\frac{\partial L}{\partial X}-D_\tau\big(\frac{\partial L}{\partial D_\tau X}\big)\Big)\delta X_\tau d\tau+E_{qt}\Big[\big(\frac{\partial L}{\partial D_\tau X}\big)+\nabla S_u)\delta X_u\Big]=0$$

For our L,

$\Rightarrow$ Stochastic (almost sure) Euler-Lagrange :

$(EL)\quad D_\tau D_\tau X=\nabla V(X)\,,t<\tau<u\,,$ with BC $X_t=q,\ D_uX=-\nabla S_u(X_u)$

By def of D_τ , taking absolute expectation $E[...]$,

$$\frac{d^2}{d\tau^2}E[X_\tau]=E[\nabla V(X_\tau)] \qquad \text{Dynamical law}$$

(in analogy with Feynman)

What about the dual dynamics of X_τ on $[s,u]$ in terms of $\mathcal{F}_t$?

Since $D_\tau \to -D_\tau^*$ under TR on $[s,u]$ (Proof: if $\hat{X}(t) = X(u+s-t)$ compute $D_t\hat{X}$) , for our H

$$J^*[X(\cdot), t-s] = E_{qt}\{\int_s^t \left(\frac{1}{2}|D_\tau^* X|^2 + V(X_\tau)\right) d\tau + S_s^*(X_s)\}$$

whose $\delta J^*[X](\delta X) = 0$ is associated with dual HJB solution

$$(HJB)^* \quad \frac{\partial S_L^*}{\partial t} + \frac{1}{2}|\nabla S_L^*|^2 - \frac{\hbar}{2}\Delta S_L - V = 0 \qquad \text{with} \;\; S_L^*(q,s) = S_s^*(q)$$

and minimizing drift $\quad D_\tau^* X = B^*(X,\tau) = \nabla S_L^*(X,\tau), \; \tau \in [s,t]$

Remarks :

a) As a deterministic PDE, $\nabla_q(HJB) = EL$ (Cf Integrable classical systems)

b) Probabilistic (but Euclidean) version of Feynman's commutation relation :

$$E[X_j D_t^* X^k - D_t X^k X_j] = \hbar \delta_{jk}$$

c) Time reversed $(EL) = (EL)^* = \nabla(HJB)^*$

d) Algebraic relation between drifts of optimal diffusion:

$$B^*(q,t) = B(q,t) - \hbar \nabla \log \rho_t(q)$$

For $\hbar = 0$, $D_t^* X = D_t X = \frac{d}{dt} X$

The boundary conditions BC of HJB Eqs “solve" Schrödinger’s problem (JCZ-1984-6):

For $M_M(B_s \times B_u)$ pick $\eta_s^*(x) = e^{-\frac{S_s^*(x)}{\hbar}}$, $\eta_u(z) = e^{-\frac{S_u(z)}{\hbar}}$ and take both marginals, for any given probab. densities (ρ_s, ρ_u) :

$$\begin{cases} \eta_s^*(x) \int e^{-\frac{1}{\hbar}(u-s)H}(x,z)\eta_u(z)dz = \rho_s(x) \\ \eta_u(z) \int \eta_s^*(x) e^{-\frac{1}{\hbar}(u-s)H}(x,z)dx = \rho_u(z) \end{cases}$$

System of eqs for $(\eta_s^*, \eta_u) = BC$ of 2 adjoint heat eqs.

Theorem (Beurling 1960)

For any ρ_s, ρ_u without zeroes there is existence and uniqueness of positive, not necessarily integrable solutions (η_s^*, η_u) and therefore of associated Bernstein diffusion $X_t,\ s \leq t \leq u$, with probability density $\rho_t(q)dq = \eta_t^*(q)\eta_t(q)dq$.

Symmetries of underlying random mechanics : Noether's Theorem (T-Z 1997)

Fix H as before, i.e $L(X, D_tX) = \frac{1}{2}|D_tX|^2 + V(X_t)$. Define a generator of heat symmetry group underlying X_t by

$$N = T(t)\frac{\partial}{\partial t} + Q_i(q,t)\frac{\partial}{\partial q^i} - \frac{1}{\hbar}\Phi(q,t),$$

for $\forall T, Q, \Phi$ analytical solutions of “determining ODEs" of this group i.e, s.t. $\hat{H}\eta \equiv \left(\hbar\frac{\partial}{\partial t} - H\right)\eta = 0 \Rightarrow \hat{H}N\,\eta = 0$. If L satisfies the regularized (divergence) symmetry :

$$T\frac{\partial L}{\partial t} + Q_i\frac{\partial L}{\partial q^i} + (D_tQ_i + D_tX_i\frac{dT}{dt})\frac{\partial L}{\partial D_tX_i} + L\frac{dT}{dt} = D_t\Phi$$

then, along any diffusion $X.$ extremal of Action $J[X(\cdot)]$,

$$D_t\big(\underbrace{Q_iD_tX_i - hT - \Phi}_{\mathcal{P}_t-\text{martingale}}\big)(X_t,t) = 0\ , \quad \text{for } h = \tfrac{1}{2}|B|^2 + \tfrac{\hbar}{2}\nabla.B - V$$

and analogous for $\mathcal{F}_t$-martingale.

The theory founded on the solution of Schrödinger's problem (1984-86) provides a rigorous probabilistic interpretation of all formal time discretized results of Feynman's functional calculus.

Examples:

(1) Feynman observes that the average kinetic energy for H_0 has to be $\frac{1}{2} < BB^* >_{S_L}$ and not $\frac{1}{2} < B^2 >_{S_L}$ (p.36 Feynman's Thesis, L. Brown, World Scientific 2005)

In Schrödinger's approach $E\left[\frac{1}{2}BB^*\right] = \frac{1}{2}E[B^2 + \hbar\nabla.B]$ using $B^* = B - \hbar\nabla \log \rho$ and integ. by parts

$\Rightarrow\ h = \frac{1}{2}(B^2 + \hbar\nabla.B) =$ Energy $\mathcal{P}_t$-martingale $\Rightarrow$ strong conservation law since

$\frac{d}{dt}E[\frac{1}{2}(B^2 + \hbar\nabla.B)] = 0$

(2) Brownian as a Bernstein : (Already in Schrödinger 1931 but only for $\mathcal{P}_t$)

$V = 0$ in L. For $\rho_s(x) = \delta_x$, $\rho_u(z) = h_0(x, u - s, z)$,
BC of Schrödinger $\Rightarrow \forall t \in [s, u]$, $\eta_t^*(q) = h_0(x, t - s, q)$, $\eta_t(q) = 1$ (not integrable!)

$$D_t X = -\nabla S_L(q, t) = -\hbar \nabla \log \eta_t(q) = 0$$

$$D_t^* X = \nabla S_L^*(q, t) = -\hbar \nabla \log \eta_t^*(q) = \frac{q - x}{t - s}$$

NB: $\Delta S_L^* = \frac{1}{t-s}$ without effect $\Rightarrow$ Classical velocity OK here. In general the drifts are $\hbar$-dependent.

$X_t = W_t$ irreversible ! But switching boundary probabilities :

$\rho_s(x) = h_0(x, u - s, z)$, $\rho_u(z) = \delta_z$ get another $\hat{X}_t$ with symmetric look.

Finally, if $\rho_s(x) = \delta_x,\ \rho_u(z) = \delta_z$,

$$\eta_t^*(q) = h_0(x, t-s, q)\ , \quad \eta_t(q) = h_0(q, u-t, z)$$

$X_t =$ "Brownian Bridge", backbone $(\eta_s^* = \delta_x, \eta_u = \delta_z)$ of all "free" $(V = 0)$ Markovian Bernstein diffusions, with arbitrary $(\eta_s^*, \eta_u) > 0$.

For Noether , and $H = H_0 + V,\ \ \forall\, L = \frac{1}{2}|D_t X|^2 + V(X)$, i.e V time independent, admits symmetry

$T = 1,\ Q = \Phi = 0 \Rightarrow h = \frac{1}{2}B^2 + \hbar\nabla.B + V = \mathcal{P}_t$ energy martingale

$\longrightarrow E[h(X_t, t)] = \text{cste}.$

(3) Schrödinger's problem was a classical statistical analogy with pure quantum states, involving Markovian Bernstein.

Mixed quantum states correspond to joint measure $M(dx, dz)$ which are not of Feynman-like Markovian form.

Ex: Periodic Ornstein-Ulhenbeck

$$\lambda > 0, \ \ X_t = X_0 - \lambda \int_0^t X_\tau d\tau \ \ , \ \lambda > 0, \ \ t \in [0, 1] \ \ \underline{\text{with} \ \ X_0 = X_1} \ \ a.s$$

NB: As long as $X_0 = X_1$ are deterministic points, X_t is still Markov.
(Cf P. Vuillemot, JCZ (2020))

3. Relations with Mass Transportation Theory (or OT)

There, Feynman's trajectorial view of stochastic processes is replaced by analytical approach of their measures.

Measure theoretic reformulation of Schrödinger's problem:

Def: Relative entropy of a probab. measure P w.r.t another R :

$$H(P|R) = \int_\Omega \ln\left(\frac{dP}{dR}\right) dP$$

<u>Problem:</u> Minimize $P \to H(P|R)$ among all P on $\Omega = C([s,u],\mathbb{R}^n)$ s.t P_s and P_u are fixed and R is the measure of Markovian Bernstein X_t with $D_t X_t = D_t^* X_t = 0$ (regrettably called, sometimes, “reversible" Brownian)

Solution : $\hat{P} = \eta_s^*(X_s) R\, \eta_u(X_u)$

Original Schrödinger's case : $H = H_0$

Disintegrate optimal $\hat{P}$ into its backbone bridges :

$$\hat{P}(\cdot) = \int R^{xz}(\cdot)\hat{M} = \int_{\mathbb{R}^{2n}} h_0(x, u-s, z)\hat{M}(dx, dz)$$

Marginals of $R^{xz} \rightarrow$ Solution $(\eta_s^*, \eta_u) = (1, 1)$.
$\hat{M}$ itself solves the problem :

Minimize $H(M|R^{xz})$ among all joint measures $M(dx, dz)$ with marginals

$$M(\mathbb{R}^n, dz) = P_u = \rho_u(dz) \ , \quad M(dx, \mathbb{R}^n) = P_s = \rho_s(dx)$$

Similar to Monge-Kantorovich problem : (1781-1942)

Minimize $\int_{\mathbb{R}^{2n}} c(x, z) m(dx, dz)$ among all $m(dx, dz)$ with
$m(dx, \mathbb{R}^n) = \rho_s(dx), \ m(\mathbb{R}^n, dz) = \rho_u(dz)$ for given
"cost c of transportation" of a mass x to z (for ex. , with H_0) :

$c(x, z) = \frac{1}{2}|x - z|^2$ natural for us : when $\omega \in C^2$,

$$\text{Inf } \{S_L[\omega] = \int_s^u \frac{1}{2}|\dot{\omega}(t)|^2 \, dt, \ \ \omega \in \Omega_x^z\} = c(x, z)$$

In this sense, Schrödinger's method provided a regularization of Monge problem, 10 years before Kantorovich one !
For more, cf T. Mikami (2004), C. Léonard (2014)

Conclusions to take back home

Theory founded on Schrödinger's problem $\simeq$ Feynman-like regularization of classical dynamical systems $\simeq$ "Entropic regularization" of OT

1) Method independent from H : Any stochastic process can be turned time-symmetric in Schrödinger's sense (Cf N. Privault, JCZ(2004))$\longrightarrow$ Much more dynamical scenarios. Relevant to foundations of Statistical Physics.

Revisit foundations of QFT and SFT of the 70th.

2) Need a stochastic dynamical model ? Forget about traditional Cauchy probabilistic problem and adopt Schrödinger's boundary value problem: For a fixed H, pick two probability densities, and construct the optimal Bernstein process (measure) interpolating in between.

Schrödinger's miracle : the process solving his problem enjoys a kind of intrinsic time reversibility, close to quantum structures.

3) The various scientific communities involved did not speak to each other for ages. The revival of Schrödinger's problem and the theory built on it is a golden opportunity to change that.

4) Recent progress: Closely related ideas have been used recently in Hydrodynamics, from the Mass Transportation and Geometric Mechanics viewpoints. There, what plays the role of Classical $\Rightarrow$ Probabilistic regularization (in $\hbar$) is Euler-equation $\Rightarrow$ Navier-Stokes equation (in viscosity ν) .

5) Schrödinger's problem also useful in Mathematical Economy. (A. Galichon (2016))

- R. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Physics 20 (1948)
  Also L.M. Brown's *Feynman's thesis, A new approach to quantum theory*, World Scientific (2005)
- E. Schrödinger, *Sur la théorie relativiste de l'electron et l'interprétation de la mécanique quantique*, Ann. Inst. H. Poincaré 2 (1932), paragraph VII
- S. Bernstein, *Sur les liaisons entre les grandeurs aléatoires*, Verh. der Intern. Mathematikerkongr., Band 1 (Zürich, 1932)
- B. Jamison, *Reciprocal processes*, Z. Wahrsch. Gebiete 30 (1974)
- A. Beurling, *An automorphism of product measures*, Annals of Mathematics 72 (1960)
- J.C.Z., *Variational processes and stochastic versions of Mechanics*, J. Math. Phys. 27 (1986)

- A.B. Cruzeiro, JCZ, *Malliavin calculus and Euclidean quantum mechanics*, J. Funct. Anal. 96 (1991)
- M. Thieullen, JCZ, *Probabilities and quantum symmetries I. The Theorem of Noether in Schrödinger's Euclidean quantum mechanics*, Ann. Inst. H. Poincaré 67 (1997)
- T. Mikami, *Monge's problem with a quadratic cost by the zero-noise limit of h-path processes*, Prob. Theory and Related Fields 192 (2) (2014)
- C. Léonard, *A survey of Schrödinger's problem and some of its connections with optimal transport*, Discrete and Cont. Dynam. Systems, A , 34 (2014)
- L.V. Kantorovich, *On a problem of Monge*, Uspekhi Mat. Nauk. 3 (1948)

- P. Vuillermot, J.C.Z., *On Bernstein processes generated by hierarchies of linear parabolic systems in* $\mathbb{R}^d$, Stoch. Processes ant their Applic. 130 Issue 5, (2020)
- M. Arnaudon, A.B. Cruzeiro, C. Léonard, JCZ, *An entropic interpolation problem for incompressible viscous fluids*, Ann. Inst. H. Poincaré, Prob. et Stat., 56, n. 3 (2020)
- A. Galichon, *Optimal transport methods in Economics*, Princeton Univ. Press (2016)
- JCZ, *The research program of Stochastic Deformation (with a view toward Geometric Mechanics)*, in Stochastic Analysis: a series of lectures, Birkhäuser P.P. 68, Ed. R. Dalang, M. Dozzi, F. Flandoli, F. Russo (2015)
- N. Privault, JCZ *Markovian bridges and reversible diffusions with jumps*, Ann. Inst. H. Poincaré, Probab. Statist. (2004)

- C. Léonard, S. Roelly, JCZ, *Reciprocal processes. A measure-theoretical point of view*, Probab. Survey (2014)
- K.L. Chung, JCZ, *Introduction to random time and quantum randomness*, 2nd part, World Scientific (2003)