'Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning'

Ėcole de Physique des Houches
Les Houches, July, 2020.

Computational dynamics of reduced coupled multibody-fluid system in Lie group setting

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Outline

- Motivation
- Solid / fluid: geometric mechanics
- Multibody-fluid system coupled formulation
- Examples

Motivation

Why geometric approach ?

- **Valuable mathematical insight to various problems of mechanics and engineering**

  - traditional link between analytical mechanics and differential-geometry

  - modelling of finite (3D) rotations

  - numerical treatment of configuration constraints

  $\Rightarrow$ call 'naturally' for geometric mathematical framework

- **Design of structure-preserving numerical integration methods**

  a) 'Smart' integration procedures that respect underlying kinematic and dynamic structure of the system $\rightarrow$ qualitative behaviour

  - integration methods on manifolds and Lie-groups

- **Non-linear control design**
- **Connections to other fields of physics and engineering**

  - mechanics of continuous media /multi-physics

  $\rightarrow$ fluid mechanics

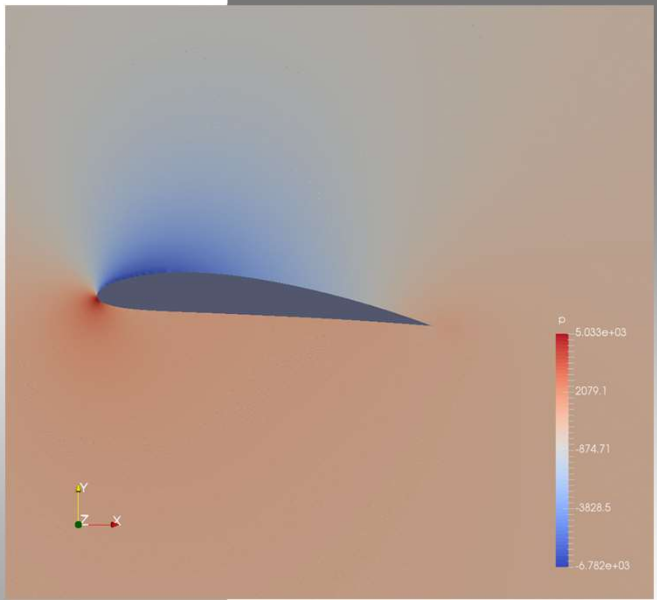
  / fluid-structure interaction

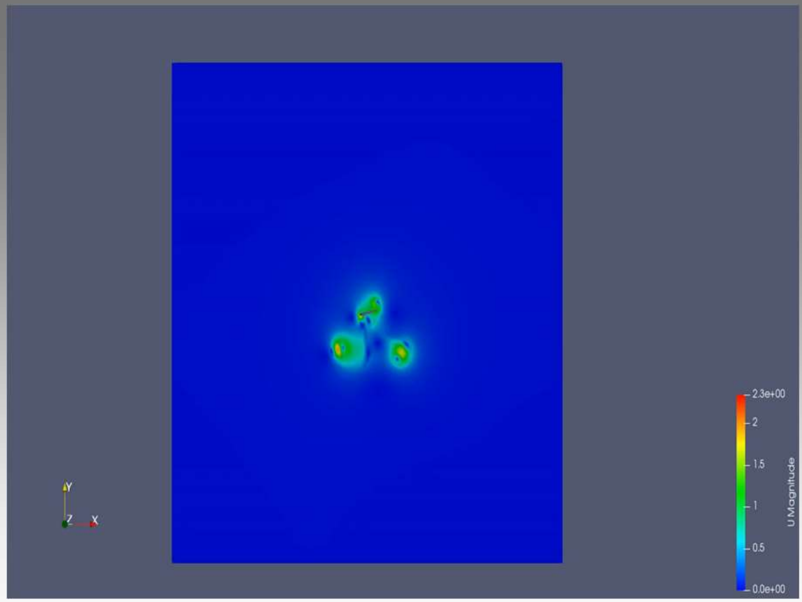
  $\rightarrow$ magnetism, optics

  $\rightarrow$ nano-scale systems

Motivation / Fluid-structure interaction

- **discretisation of fluid domain + coupling with solid motion**
- **loosely coupled algorithms / ALE formulation**
- **moving FV mesh: problems with accuracy, stability...**





- **meshless methods: SPH, Lattice-Boltzmann Method ...**

Geometric mechanics approach to the coupled problem

- **Geometric mechanics framework:**
  - **rigid bodies and ideal incompressible fluid**
    - **analogy in the mathematical structure**
    - **possibility for derivation of 'monolithic' solver**
  - **configuration space: Lie groups**
    - **rigid body** $\Rightarrow \; SE(3) \;\; (SO(3) \times \mathcal{R}^3); \;\; SE(2)$
    - **incompressible fluid** $\Rightarrow \mathrm{Diff}_{\mathrm{Vol}}(\mathcal{F})$

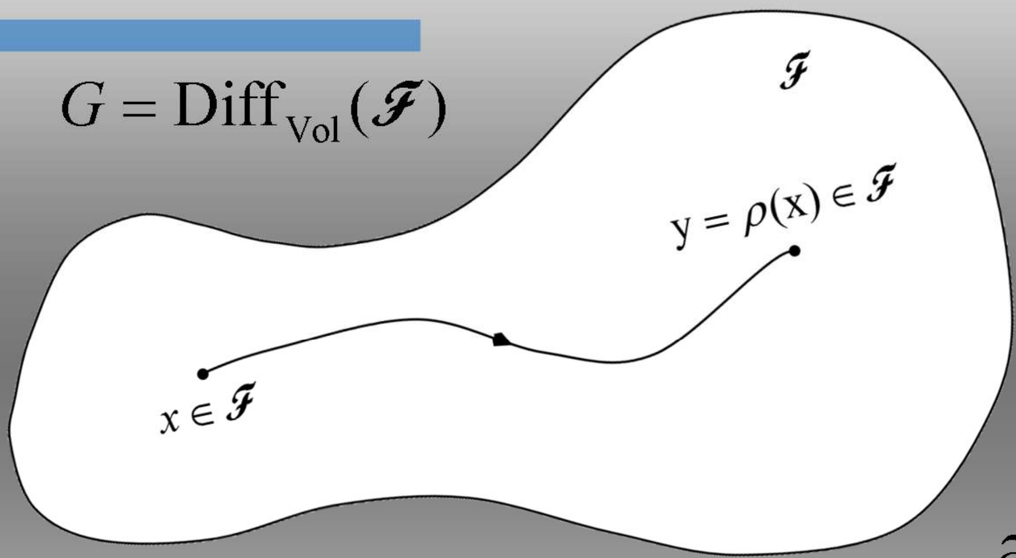
$$\mathcal{F}: \text{domain in } \mathcal{R}^2 \text{ or } \mathcal{R}^3$$

$$\text{(infinite-dimensional group)}$$

Arnold, V.: *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits.* Annales de l'Institut Fourier 16(1), 319-361 (1966)

Incompressible ideal fluid

$$G = \mathrm{Diff}_{\mathrm{Vol}}(\mathcal{F})$$



- **Material velocity field (Lagrangian):** $V(x,t) = \dfrac{\partial \rho(x,t)}{\partial t}$
- **Spatial velocities (Eulerian):** $\mathbf{v}(y,t) = \dot{\rho} \circ \rho^{-1}$

$\Rightarrow \mathbf{v} = \dot{\rho} \circ \rho^{-1}$ **independent of** η_{fixed}

$$\rho \mapsto \rho \circ \eta_{fixed}$$

(right invariance of Eulerian description)

Lie algebra and dual spaces

- **Velocities**
  - **at the identity: Lie algebras**

    rigid body $\Rightarrow se(3) \ \ (so(3)\times \mathcal{R}^3)\ ;\ se(2)$

    $$\tilde{\omega} \in so(3)$$

  - **incompressible fluid** $\Rightarrow \mathrm{V}_{\mathrm{Div}}(\mathcal{F}_0)$

    divergence-free vector fields in $\mathcal{F}_0$

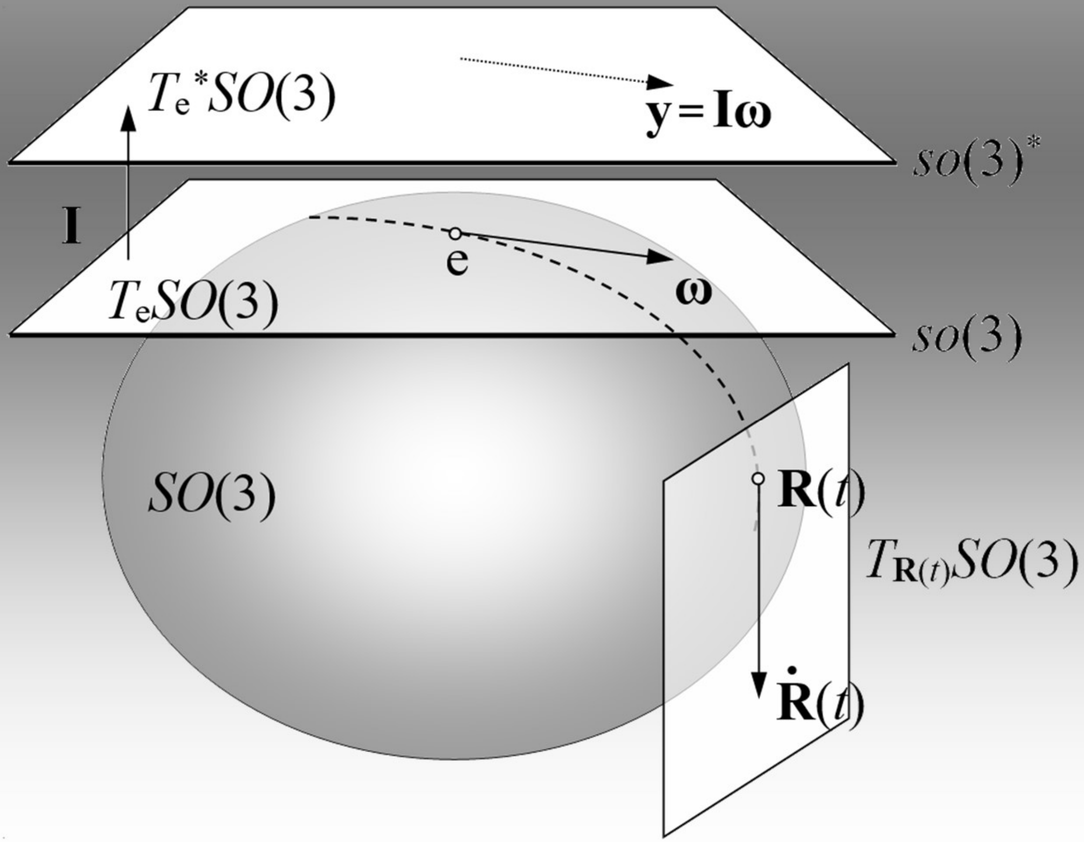
    tangent to the boundary $\partial \mathcal{F}_0$

- **Dual spaces of Lie algebras**
  - **rigid body** $\Rightarrow \ \ y = I\omega \in so(3)^*$ **... angular momentum**
  - **incompresible fluid** $\Rightarrow$ **vorticity field (circulation)** $\mathrm{V}^*_{\mathrm{Div}}(\mathcal{F}_0)$

    for a flow $\ \mathbf{v} \in \mathrm{V}_{\mathrm{Div}}(\mathcal{F}_0)$

Example: $SO(3)$ / tangent and cotangent space

- **angular velocity 'ω' & angular momentum 'y':**



Dynamics

- **Rotation motion of rigid body**
  - **geodesic on $G = SO(3)$ equipped with left-invariant Riemannian metric**
- **Motion of ideal incompressible fluid**
  - **geodesic on $G = \mathrm{Diff}_{\mathrm{Vol}}(\mathscr{F})$ with right-invariant Riemannian metric**
- **Dynamics evolves on:**

TG : **'positions' + velocities *(Lie algebra)*: body, fluid**

T^*G: **'positions' + momentum (body) / vorticity (fluid)**

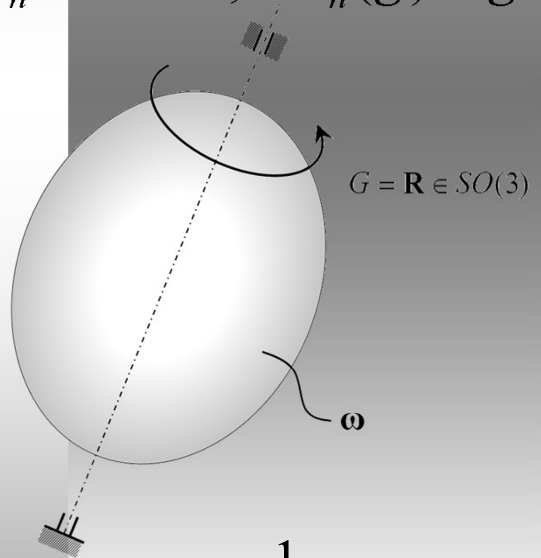
(dual Lie algebra)

Symmetries

Rigid body

$$G = SO(3)$$

$$L_h : G \to G; \quad L_h(g) : hg$$

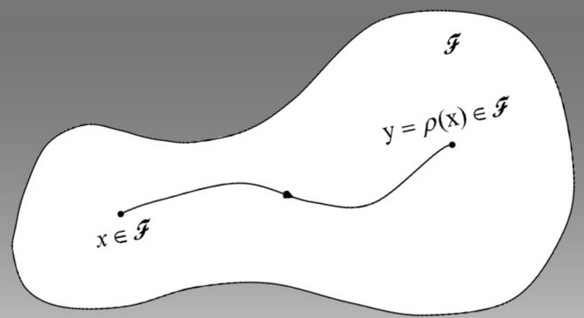


$$KE_{body} = \frac{1}{2}\omega^T I\omega$$

Incompressible fluid

$$G = \mathrm{Diff}_{\mathrm{Vol}}(\mathcal{F})$$

$$R_h : G \to G; \quad R_h(g) : gh$$



$$KE_{fluid} = \frac{1}{2}\iint_{\mathcal{F}} \mathbf{v}^2 dv$$

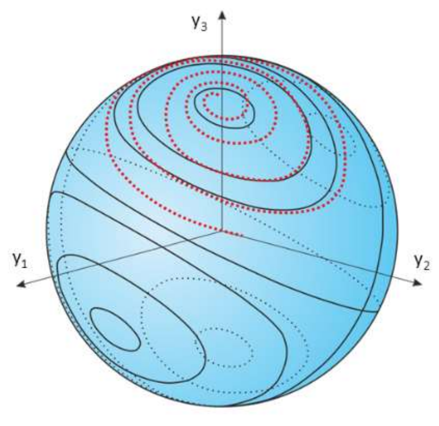
$L_h, R_h \to KE_{invariant} \xrightarrow{\textit{Noether theorem}}$ momentum map J conserved

Momentum map: $SO(3)$

- **Momentum map associated with** $G = SO(3)$

$$J_G : T^* G \to so(3)^*$$

  - **angular momentum** $y = I\omega \in so(3)^*$ **conserved**
    $\Rightarrow$ **'coadjoint orbits' preserved**



Forward Euler on Lie-group

$$\text{Euler equation : } \frac{dy}{dt} = -\omega \times y$$

- solution in the form:

$$y^{n+1} = \mathbf{R}^{\mathrm{T}}(t)\, y^n \quad , \quad y^{n+1} = \mathrm{Ad}_G^* y^n$$

$$G = \mathbf{R} \in SO(3)$$

  - ***KE*** **and Lagrangian:** G **- invariant**

Momentum map: $\mathrm{Diff}_{\mathrm{Vol}}(\mathscr{F})$

- **Momentum map associated with** $\mathrm{Diff}_{\mathrm{Vol}}(\mathscr{F})$

$$J_{\mathrm{Diff}_{\mathrm{Vol}}} : T^{*}\mathrm{Diff}_{\mathrm{Vol}} \to V^{*}_{\mathrm{Div}}(\mathscr{F}_{\mathrm{o}})$$

- **vorticity advection /** $V^{*}_{\mathrm{Div}}(\mathscr{F}_{\mathrm{o}})$ **: vorticity + circulation**

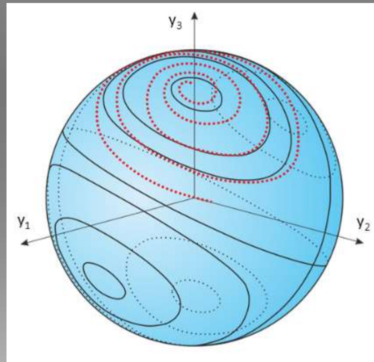
$$[\alpha] \in V^{*}_{\mathrm{Div}}(\mathscr{F}_{\mathrm{o}}) \; ; \; [\alpha] \mapsto (\mathbf{d}\alpha, \Gamma := \int_{\mathrm{c}} \alpha)$$

- **fluid *KE* and Lagrangian** $\Rightarrow$ $\mathrm{Diff}_{\mathrm{Vol}}$ **- invariant**
  - **momentum map is conserved**
  - **particle relabeling symmetry**
  - **Kelvin's theorem: circulation around closed curve -> preserved; vorticity is advected**
    - **coadjoint orbits preserved !!**

Euler equations: body rotation / fluid dynamics

- **Second Euler teorem on arbitrary Lie group G (left invariant metric):**

$$\frac{d\mu}{dt} = ad^*_{\omega_{Lie}}\mu \; ; \; \mu \in g^*$$



Forward Euler on Lie-group

  - **rigid body:** $\mu = y \in so(3)^*$

$$\dot{y} = ad^*_{\omega} y = \text{-}\omega \times y \quad \Rightarrow \quad y = \mathrm{Ad}^*_G y$$

  - **ideal fluid:** $[\alpha] \in V^*_{\mathrm{Div}}(\mathscr{F}_{\mathrm{o}}) \; ; \; [\alpha] \mapsto (\mathbf{d}\alpha, \Gamma)$

$$\dot{v} = -\mathrm{B}(\mathbf{v},\mathbf{v}) \;\; ; \;\; \mathrm{B}(c,a) = \mathrm{curl}c \times a + \mathrm{grad}p$$

$$(\text{in } \mathscr{R}^3)\ldots \quad \frac{\partial \mathbf{v}}{\partial t} = v \times \mathrm{curl}v - \mathrm{grad}p$$

  - **solution: preservation of coadjoint orbits**

    - body angular momentum; fluid circulation:

$$\mathrm{Ad}^*_{\mathrm{Diff}_{\mathrm{Vol}}}(\mathbf{d}\alpha, \Gamma) = (\mathbf{d}(\mathrm{Diff}_{\mathrm{Vol}*}\alpha), \Gamma)$$

Monolithic coupling

- **Geometric Fluid-MBS monolithic coupling**

$$\mathcal{M} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k \cup \mathcal{F}$$

- **Assumptions:**
  - ✓ **potential flow**
  - ✓ **no vorticity (no circulation!)**
  - ✓ $\mathcal{F}$ **is connected**
    - **fluid velocity field:** $\mathbf{v} = \nabla \phi$

      : incompressibility $\Rightarrow \Delta\phi = 0$ **in** $\mathcal{F}$
- **Configuration space:** $Q(\mathcal{M}) := \{ f : \mathcal{M} \to \mathcal{M} \}$

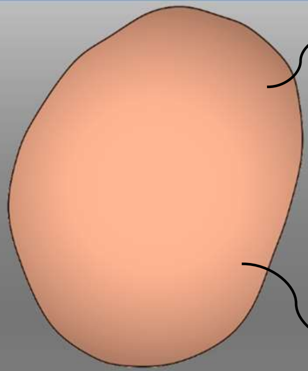
$$q_{\mathcal{F}} \in \mathrm{Diff}_{\mathrm{Vol}}(\mathcal{F})$$

q_{β_i} **~ rigid body motion** β_i

$$G_{body} \subseteq SE(3) \times \ldots \times SE(3) \,/\, \mathcal{R}^3 \times SO(3) \times \ldots \times \mathcal{R}^3 \times SO(3)$$

Fluid-Body model reduction / particle symmetry

- **Rigid body**



$$G = \mathbf{R} \in SO(3)$$

$$SO(3) \xrightarrow[(Euler-Poincare)]{Lie-Poisson}$$

$$L(\mathbf{R},\dot{\mathbf{R}}) = tr((\mathbf{R}^{-1}\dot{\mathbf{R}})^T I\, \mathbf{R}^{-1}\dot{\mathbf{R}}) \to \frac{1}{2}\omega^T I \omega$$

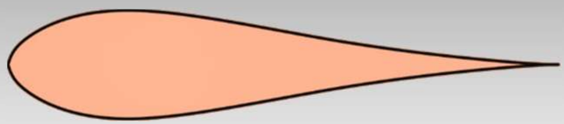
$$\omega$$

- **Euler equations in Lie-Poisson form:**

$$\dot{y} = -\omega \times y, \quad y = I\omega \in so(3)^* \qquad (I\dot{\omega} = I\omega \times \omega)$$

$$(no\ \mathbf{R}\ !!)$$

- **Rigid body + ideal fluid / particle relabeling symmetry**



$$\xleftarrow[(right)]{Symplectic\ (\Gamma=0)} \mathrm{Diff}_{\mathrm{Vol}}(\mathcal{F})$$

$$\left[(q_{\beta_i}, q_{\mathcal{F}}, y_{\beta_i}, \alpha)\right] \mapsto (q_{\beta_i}, \boldsymbol{m}(y_{\beta_i}))$$

$$T^*Q \to T^*SE(3)$$

$$(no\ fluid\ \ variables!!)$$

Fluid-MBS model reduction / particle symmetry

- **Reduction by the fluid $\mathrm{Diff}_{\mathrm{Vol}}$ symmetry:**
  - **particle relabeling symmetry $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$**
    - **fluid: Kelvin's theorem -> circulation preserved**
    - **bodies: do not affect dynamics of $\mathcal{B}_i$**
    - **.... reduction at zero circulation:**

$$(T^*Q)_{\Gamma=0} = \mathcal{J}^{-1}(O)\,/\,\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$$

  - **elimination of fluid variables** $T^*Q \to T^*G_{body}$
  - **reduced F-MBS configuration space:**
    configuration space G_{body} of the submerged solids
  - **reduced Lagrangian of F-MBS: function on TG_{body}**
  - **effect of fluid: added inertias to $\mathcal{B}_i$**

Added inertias

- **Lagrangian F-MBS:** $KE = KE_{body} + KE_{fluid}$
  - KE_{fluid} : $\mathscr{B}_i$ **added inertias ; fluid velocity** $f(\mathbf{v}_i \text{ and } \omega_i)$
- **velocity -> determined from velocity potential** ϕ

$$\Delta\phi = 0 \text{ in } \mathscr{F}$$

- with prescribed velocities at $\partial\mathscr{B}_i$

- **Neumann problem for Laplace equation *(linear !)* with boundary conditions:**

$$\nabla\phi \cdot \mathbf{n}_i = \mathbf{v}_i \cdot \mathbf{n}_i \quad \text{in } \partial\mathscr{B}_i$$

$$\nabla\phi = 0 \quad \text{at } \infty$$

- solved for translational and rotational potentials φ_i and χ_i **after transformation (by superposition) into the form:**

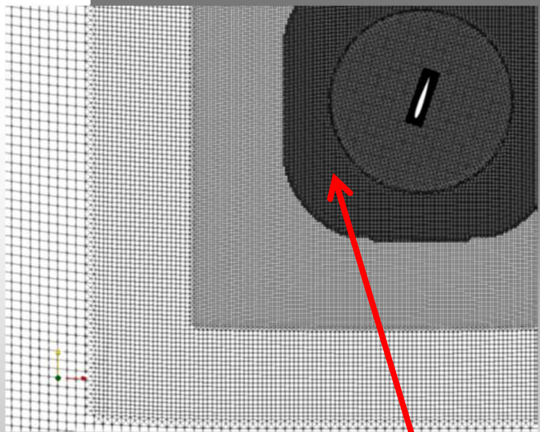
$$\phi = \sum_{i=1}^{N} (\omega_i \cdot \chi_i + \mathbf{v}_i \cdot \varphi_i)$$

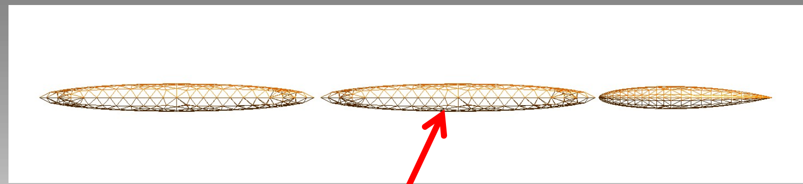
- replaced by boundary value problem -> solved via BEM

Numerical efficiency

Fluid volume discretization $\Rightarrow$ Boundary surface discretization

- **Multiple orders of magnitude fewer variables**





$$T_{\mathcal{F}} = \frac{1}{2}\rho_{\mathcal{F}}\int_{\mathcal{F}} \mathrm{div}\left(\phi\nabla\phi\right)dV = -\frac{1}{2}\rho_{\mathcal{F}}\int_{\partial\mathcal{B}} \phi\nabla\phi\cdot\mathbf{n}\,db$$

MBS configuration and state space

MBS state space:

$$S = \mathcal{R}^3 \times SO(3) \times \ldots \times \mathcal{R}^3 \times so(3) \cong TG_{body}$$

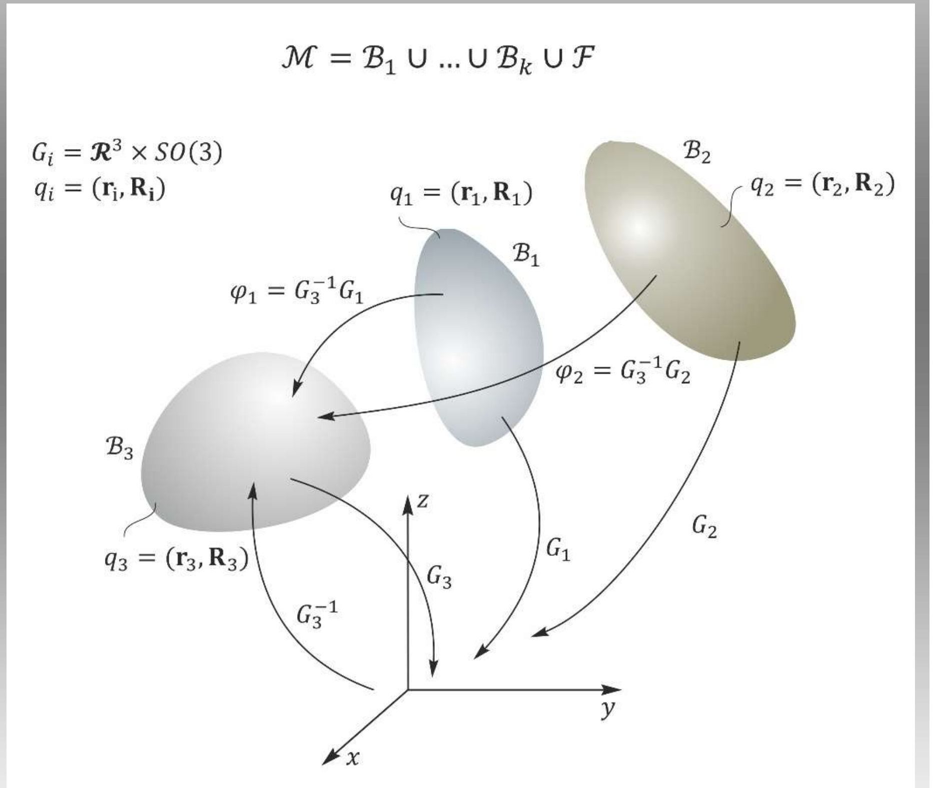
with elements:

$$x = (\mathbf{r}_1, \mathbf{R}_1, \ldots, \mathbf{r}_k, \mathbf{R}_k, \mathbf{v}_1, \tilde{\boldsymbol{\omega}}_1, \ldots, \mathbf{v}_k, \tilde{\boldsymbol{\omega}}_k)$$

- **Lie group with Lie algebra with the element**

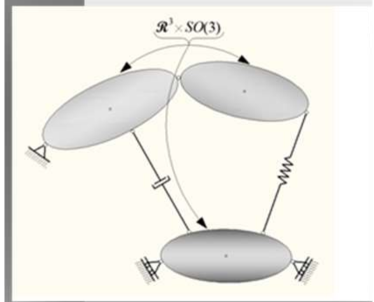
$$z = (\mathbf{v}_1, \tilde{\boldsymbol{\omega}}_1, \ldots, \mathbf{v}_k, \tilde{\boldsymbol{\omega}}_k, \dot{\mathbf{v}}_1, \dot{\boldsymbol{\omega}}_1, \ldots, \dot{\mathbf{v}}_k, \dot{\boldsymbol{\omega}}_k)$$

- **Lagrangian of F-MBS: function of x only**



Numerical integration: F-MBS system

- **Numerical integration ...** $\mathbf{M} \sim$ **added mass** ; $\mathbf{Q}=\mathbf{Q}_{ext}+\mathbf{Q}_{ideal}+\mathbf{Q}_{vort}$



$$q_0 = q_{\bar{n}-1}$$

for $i = 1, 2, \ldots, s$

$$u_i = h\sum_{j=1}^{i-1} a_{ij} f_j^*$$

$$f_i = \mathcal{F}(t_{\bar{n}-1} + c_i h, q_0 \exp(u_i))$$

$$f_i^* = \mathrm{dexp}^{-1}_{-u_i}(f_i, n)$$

end

$$v_{\bar{n}} = h\sum_{j=1}^{s} b_j f_j^*$$

$$q_{\bar{n}} = q_0 \exp(v_{\bar{n}})$$

q_n — push - forward — u_n

$$\dot{q} = L'_q(z)$$

- mapping $\mathcal{H}(t,q,Q)=\dot{\mathbf{v}}$ by solving system dynamics

$$\begin{bmatrix}\mathbf{M} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{v}} \\ \boldsymbol{\lambda}\end{bmatrix} = \begin{bmatrix}\mathbf{Q} \\ \boldsymbol{\xi}\end{bmatrix}$$

$\mathcal{F}: \mathcal{R}\times\mathcal{S}\to\mathcal{S}$ is introduced as $\mathcal{F}(t,q)=(v,\mathcal{H}(t,q))$

Lie-algebra substitution ODE
$\dot{u} = \mathrm{dexp}^{-1}_{-u}(\mathcal{F}(t,q))$, $u(0)=0$

$u_n \xrightarrow{\text{numerical method}} u_{n+1}$

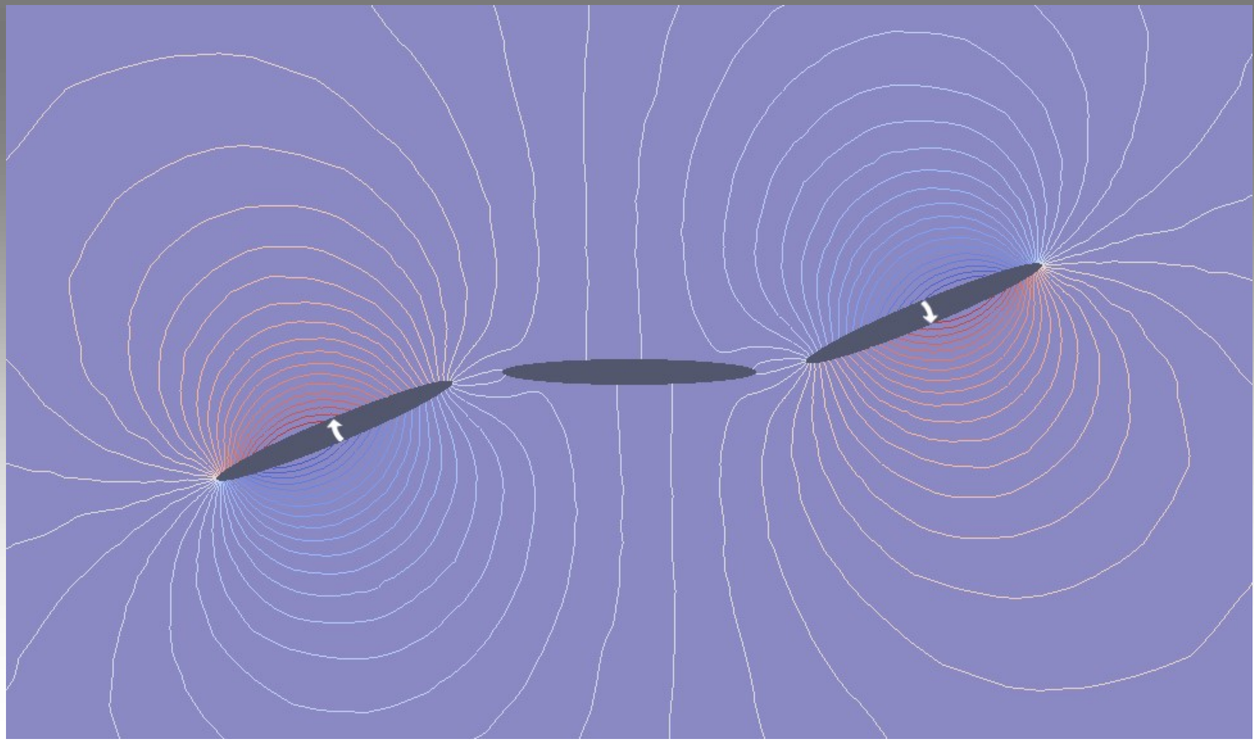
u_{n+1} — pull - back — q_{n+1}

- **Munthe-Kaas type of integration algorithm:**
- **RK scheme applied to substitution ODE in Lie-algebra** $\mathcal{S}$
- **... extended to DAE index 1 on Lie-groups**

Results (MBS + fluid flow without circulation)

Bodies with blunt edges, no circulation:

- **added masses are function of system configuration and system velocities / translations and rotations**



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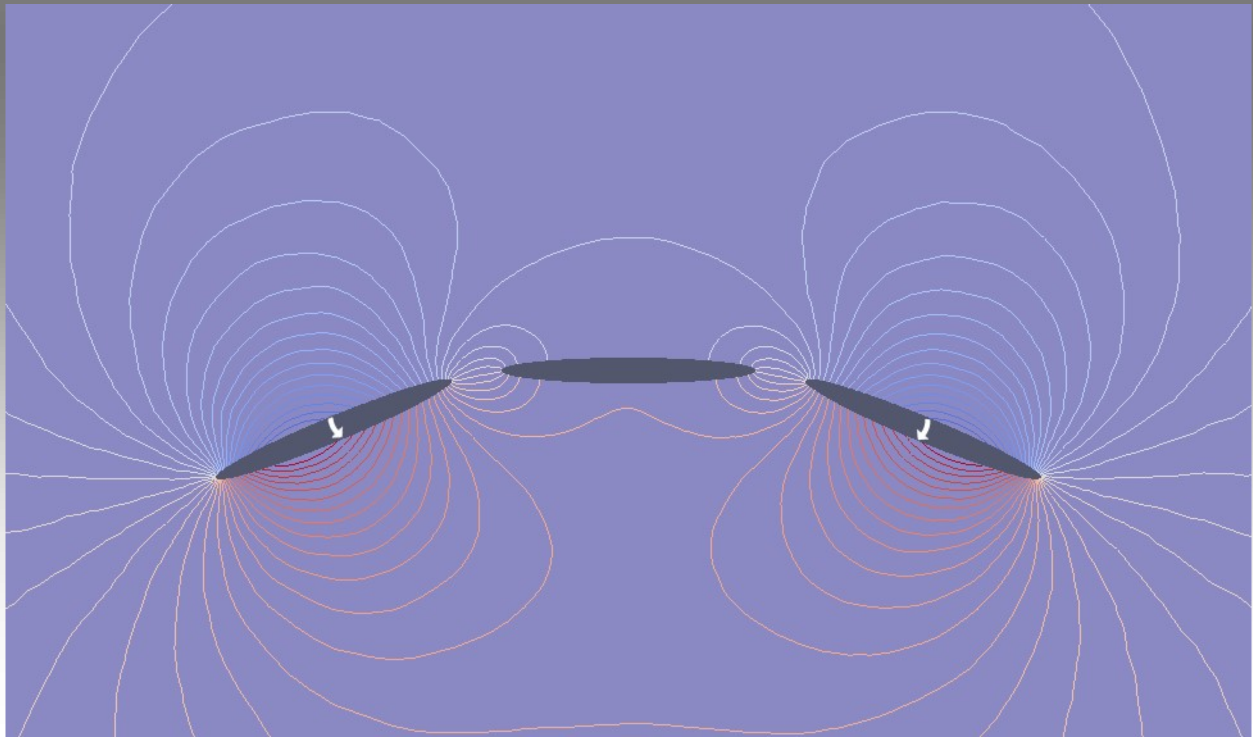

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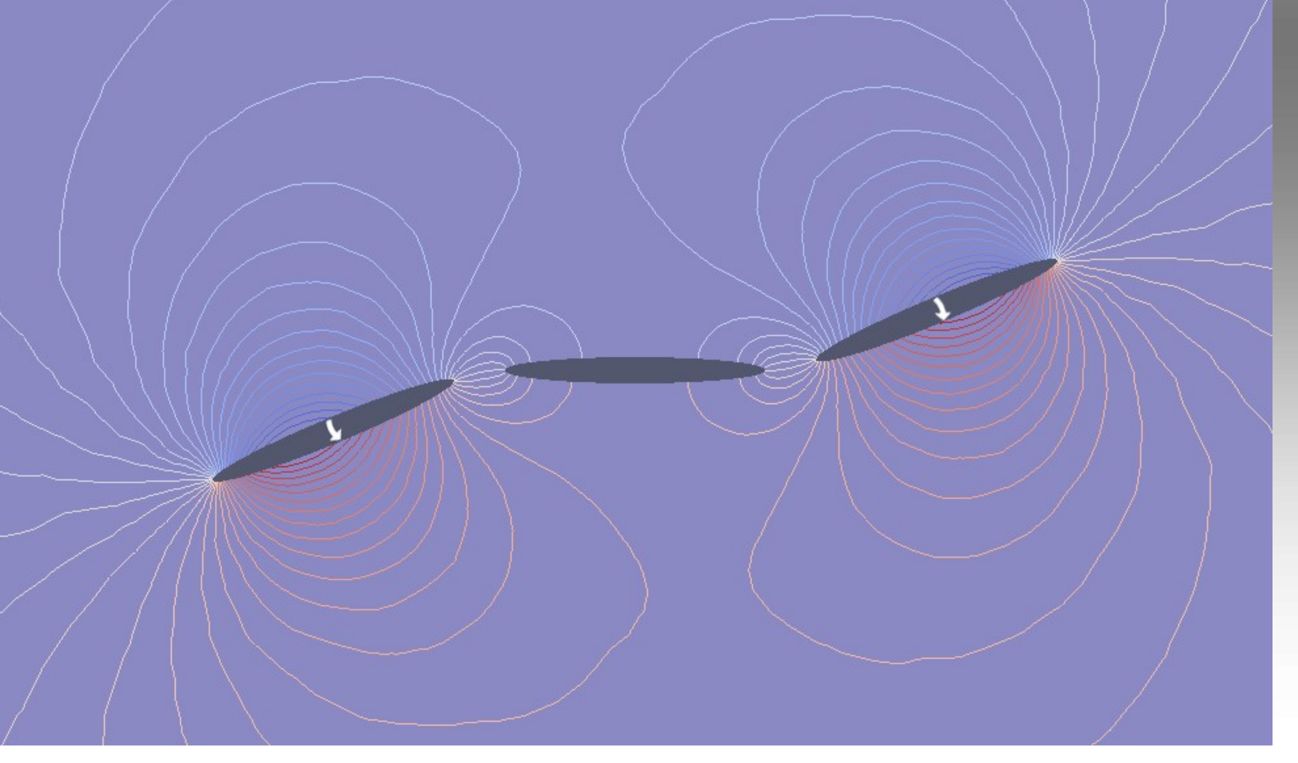

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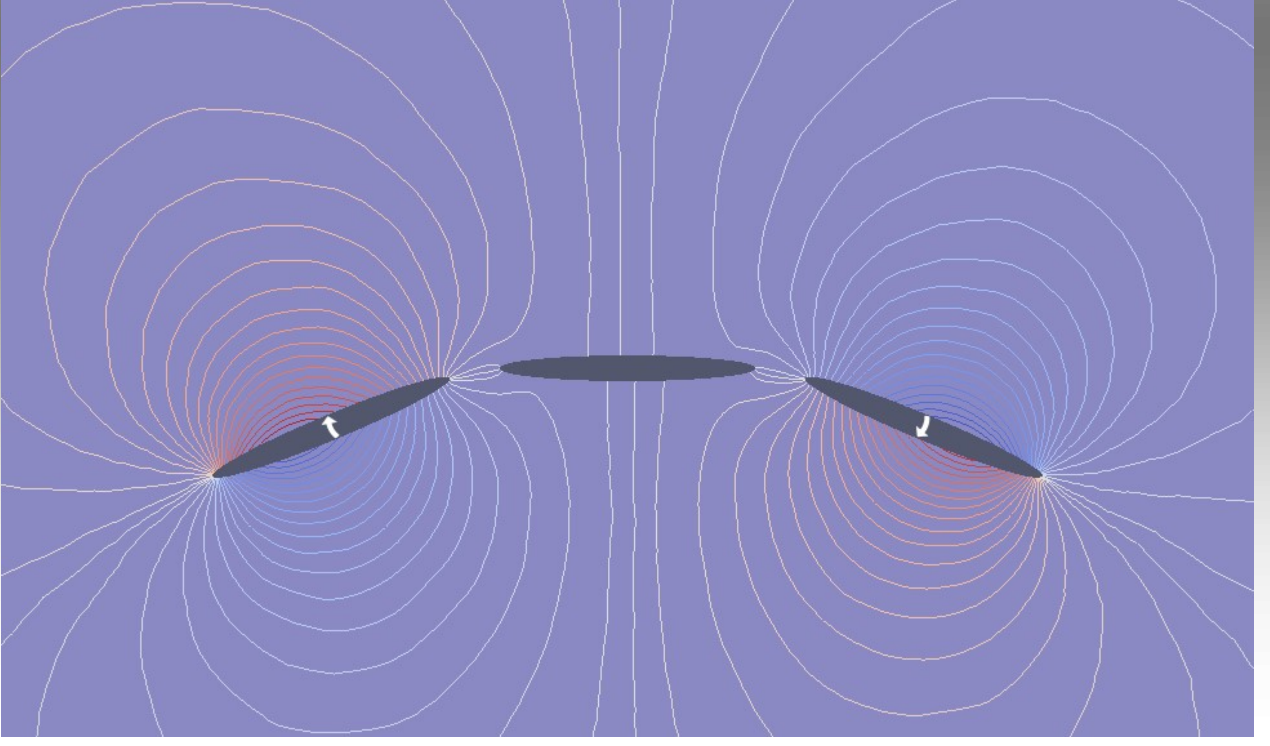


Fluid-Body / fluid flow with circulation

Marsden, J.E. et al.: *Hamiltonian Reduction by Stages.* Lecture Notes in Math., vol. 1913, Springer-Verlag Berlin Heidelberg (2007)

- **Constant circulation Γ is preserved:**

$$\mathrm{Ad}^*_{\mathrm{Diff}_{\mathrm{Vol}}}(0,\Gamma) = (0,\Gamma)$$

***(Kelvin's theorem)* ... particle relabeling symetry**

- **Fluid-body symplectic reduced space:**

$$(T^*Q)_\Gamma = \mathcal{J}^{-1}(\Gamma)/\mathrm{Diff}_{\mathrm{vol}}(\mathcal{F})$$

- **Isomorphism between $(T^*Q)_\Gamma$ and $T^*SE(2)$**

$$\left[(q_{\beta_i}, q_{\mathcal{F}}, y_{\beta_i}, \alpha)\right] \mapsto (q_{\beta_i}, \boldsymbol{m}(y_{\beta_i}))$$

  - **fluid variables $(q_{\mathcal{F}}, \alpha)$ eliminated**
    - **added mass effect encoded by $\boldsymbol{m}, y_{\beta_i} \in se(2)^*$**

$T^*Q \to T^*SE(2) \to se^*(2)$ ➢ **two stages reduction**

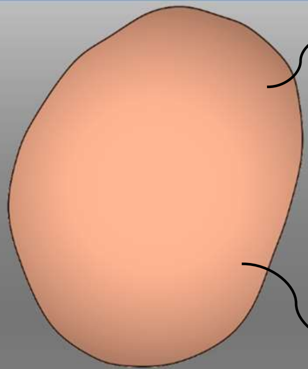
$\sim \mathrm{Diff}_{\mathrm{vol}}$ **action** $Q \to$ **principal bundle over** $SE(2)$

$\sim$ **Neumann connection** $A: TQ \to V_{\mathrm{Div}}(\mathcal{F}_0)$

-:: NC curvature: generates Kutta-Zhukowski force

Fluid - Body / two stages reduction

- **Rigid body**



$$G = \mathbf{R} \in SO(3)$$

$$SO(3) \xrightarrow[(Euler-Poincare)]{Lie-Poisson}$$

$$L(\mathbf{R}, \dot{\mathbf{R}}) = \frac{1}{2} tr(\dot{\mathbf{R}} I \dot{\mathbf{R}}^T) \rightarrow \frac{1}{2} \omega^T I \omega$$

ω

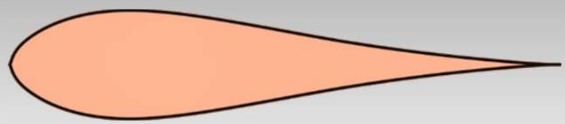
- **Euler equations in Lie-Poisson form:**

$$\dot{y} = -\omega \times y, \quad y = I\omega \in so(3)^* \quad (I\dot{\omega} = I\omega \times \omega)$$

$$(no\ \mathbf{R}\ !!)$$

- **Rigid body + ideal fluid (planar case)**

$$SE(2) \xrightarrow[(left)]{Poisson}$$



$$\xleftarrow[(right)]{Symplectic\ (\Gamma)} \mathrm{Diff}_{\mathrm{Vol}}(\mathscr{F})$$

$$T^* SE(2) \rightarrow se^*(2)$$

$$(no\ "translations"\ !!)$$

$$T^* Q \rightarrow T^* SE(2)$$

$$(no\ fluid\ variables\ !!)$$

$$\dot{\Pi} = (\mathbf{P} \times \mathbf{v}) \cdot \mathbf{b}_3$$

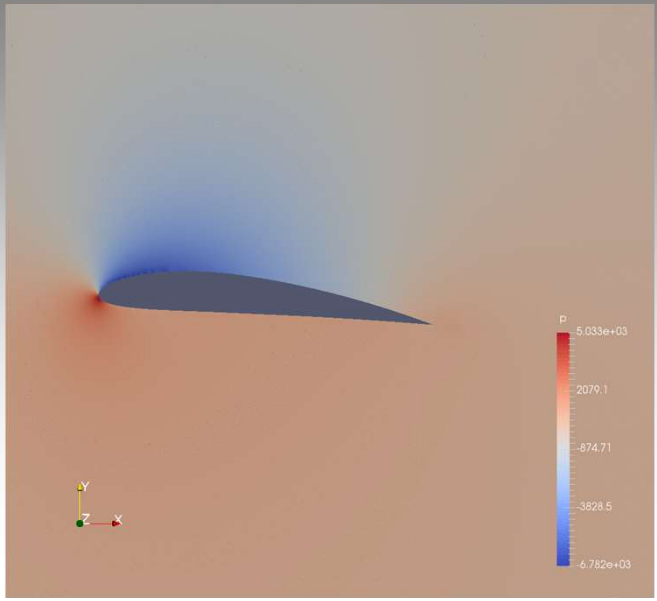
$$\dot{\mathbf{P}} = \mathbf{P} \times \omega + \Gamma \mathbf{b}_3 \times \mathbf{v}$$

$$\mathbf{P}, \Pi \in se^*(2)$$

Kutta – Zhukowski force

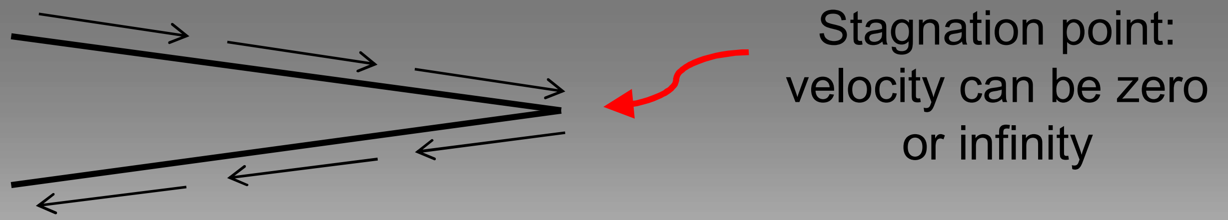
Vorticity effects? ... Oscillating airfoil

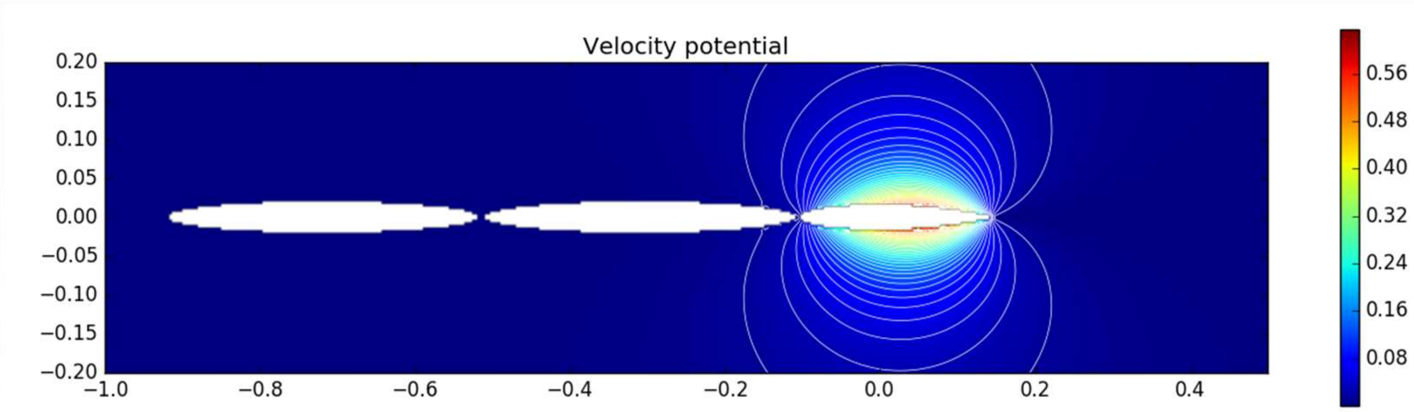
- airfoil oscillating with frequency of 100 Hz
- simulated in *foam-extend* by using RANS with k-ω turbulence model
- effect of shedding trailing edge vortices is obvious
  - vorticity effects will be accounted for by enforcing a Kutta condition



Kutta condition

- **Fluid is considered as ideal and vorticity effects are accounted for by enforcing a Kutta condition**
- **Kutta condition: velocity at the sharp edge must have meaningful physical value**

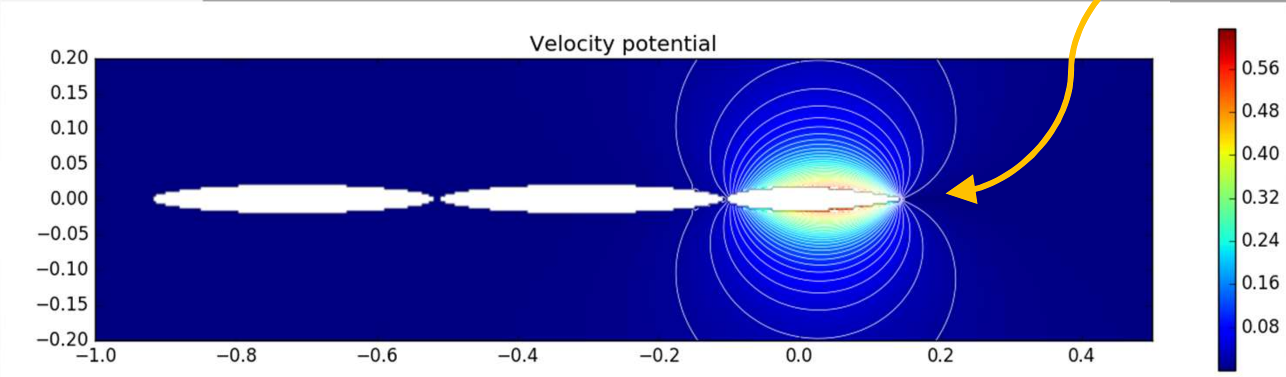




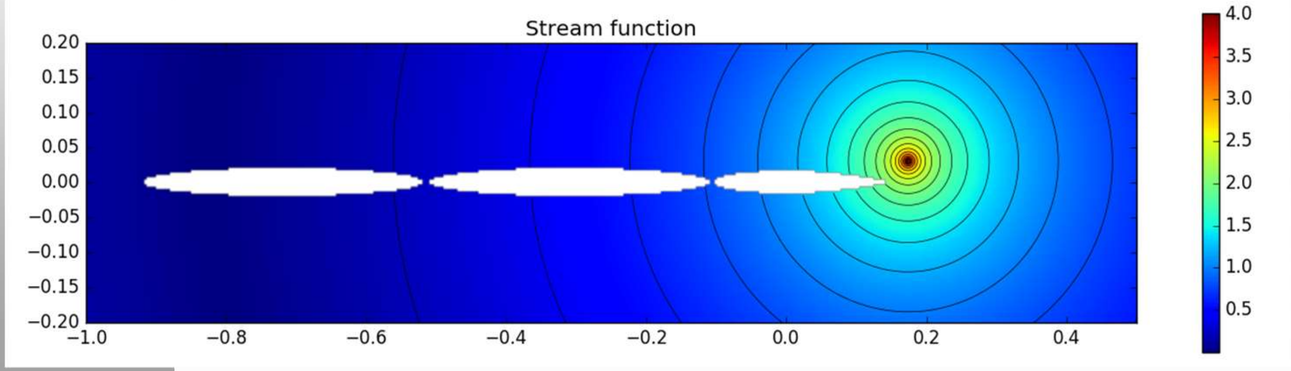
Enforcing Kutta condition

- **velocity potential…(added masses)**

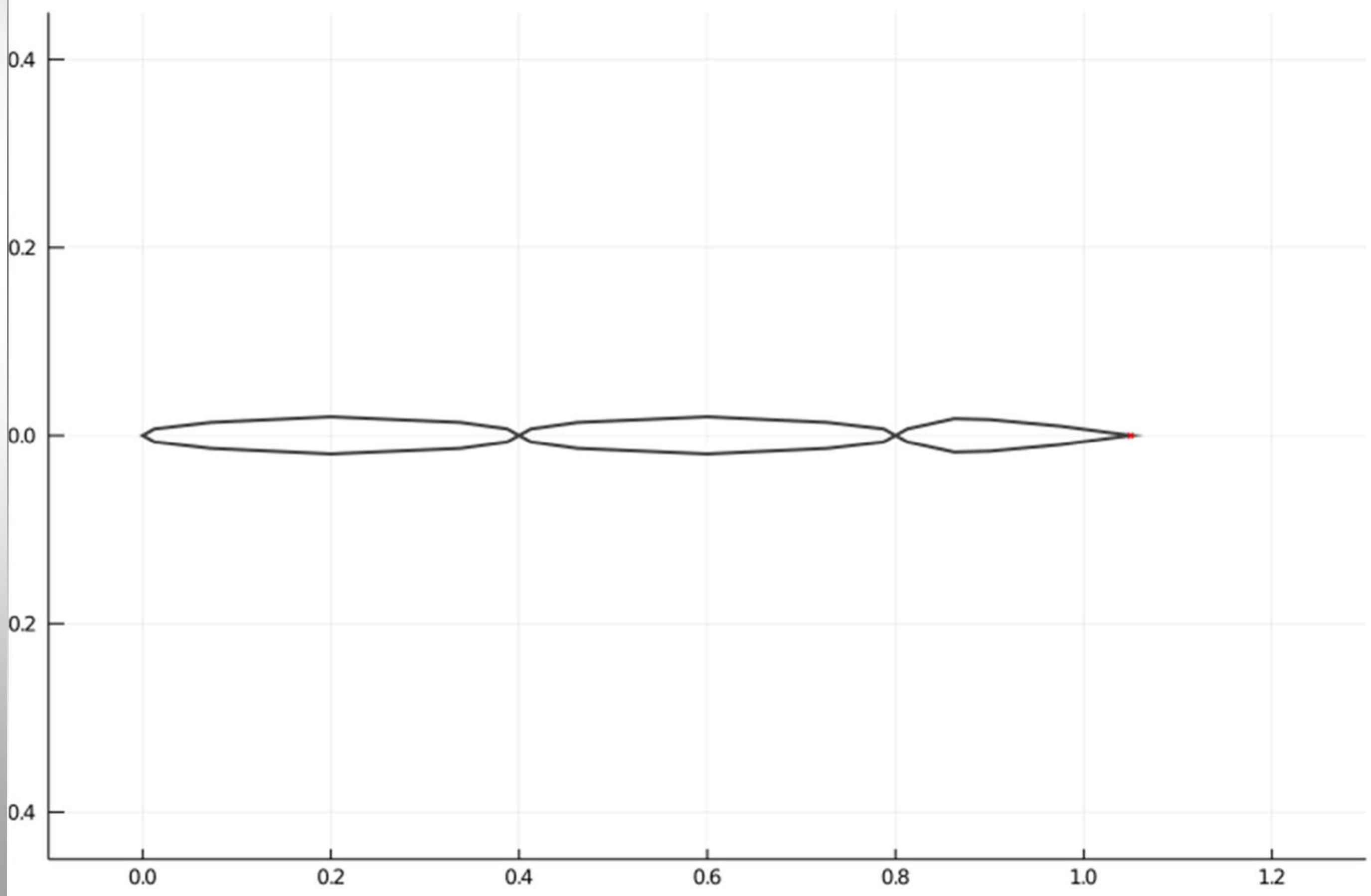
Large velocity at the sharp edge $\mathbf{v}_{\mathrm{tip}}$



- **stream function; irrotational vortex introduced** $\Gamma = 2\pi\, r\, \left|\mathbf{v}_{\mathrm{tip}}\right|$

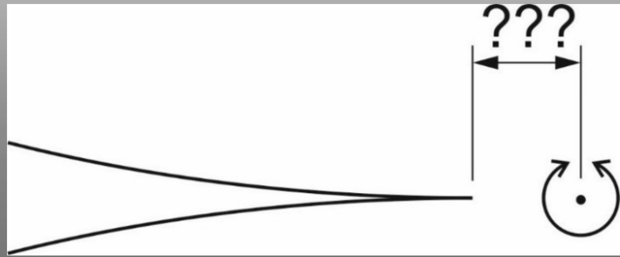


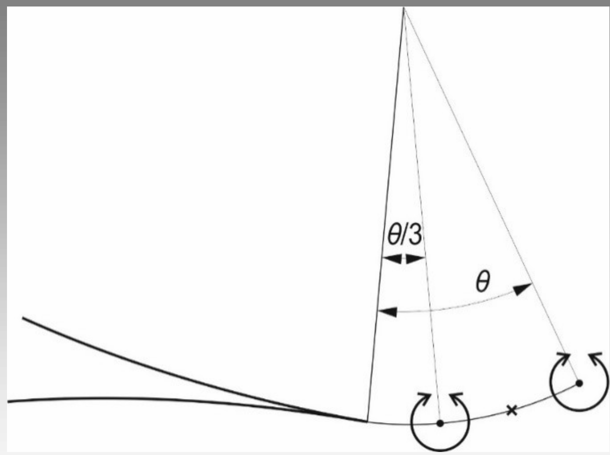
Vortex mechanism (UVLM)



Enforcing Kutta condition

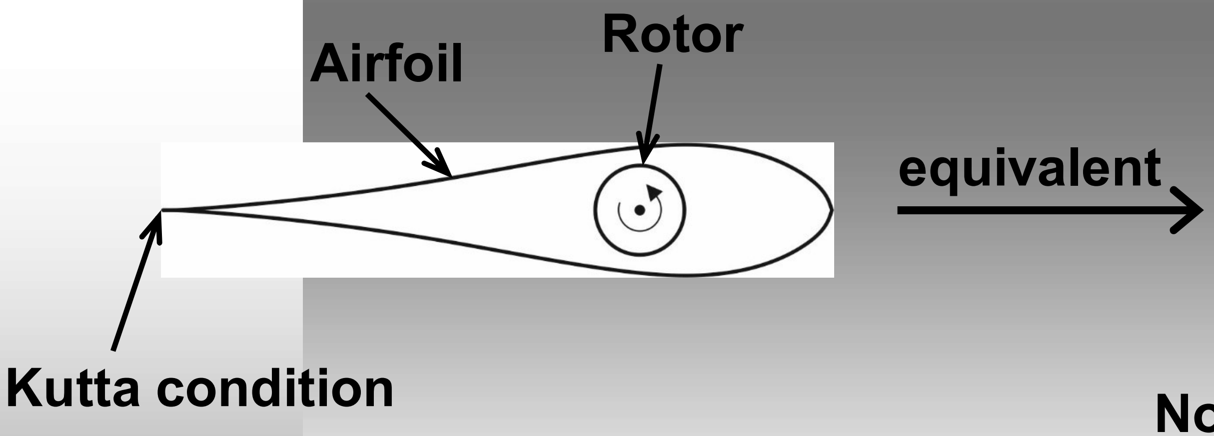
- **Kutta condition: to be enforced by shedding vortices of appropriate „strength" from the trailing edge**
- **Procedure: distance from trailing edge at which to create a new vortex at each time step?**
- **P. Tallapragada, S. Kelly [2010, 2013, 2015, 2016], Hailong X., Ph.D. thesis, 2007, Mason R.J., Ph.D. thesis, CALTECH.**

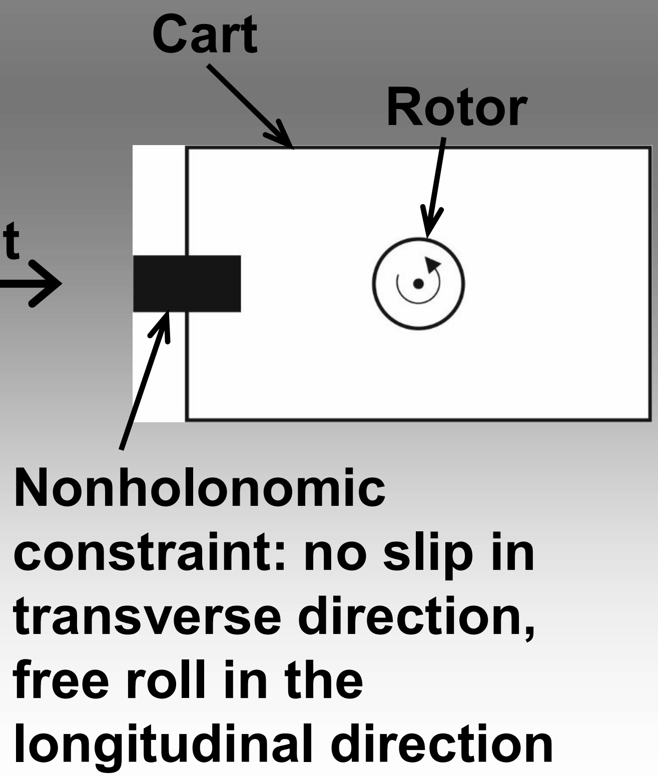




Equivalence with MBS / nonholonomic constraint

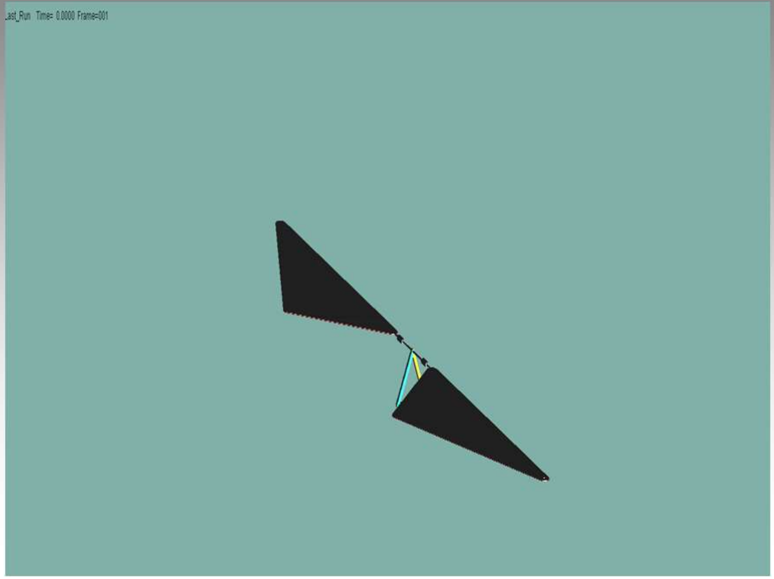
- **Kutta condition is a nonintegrable velocity constraint**
- **equivalent representation in MBS dynamics $\Rightarrow$ rear wheel constraint in the Chalpygin cart**



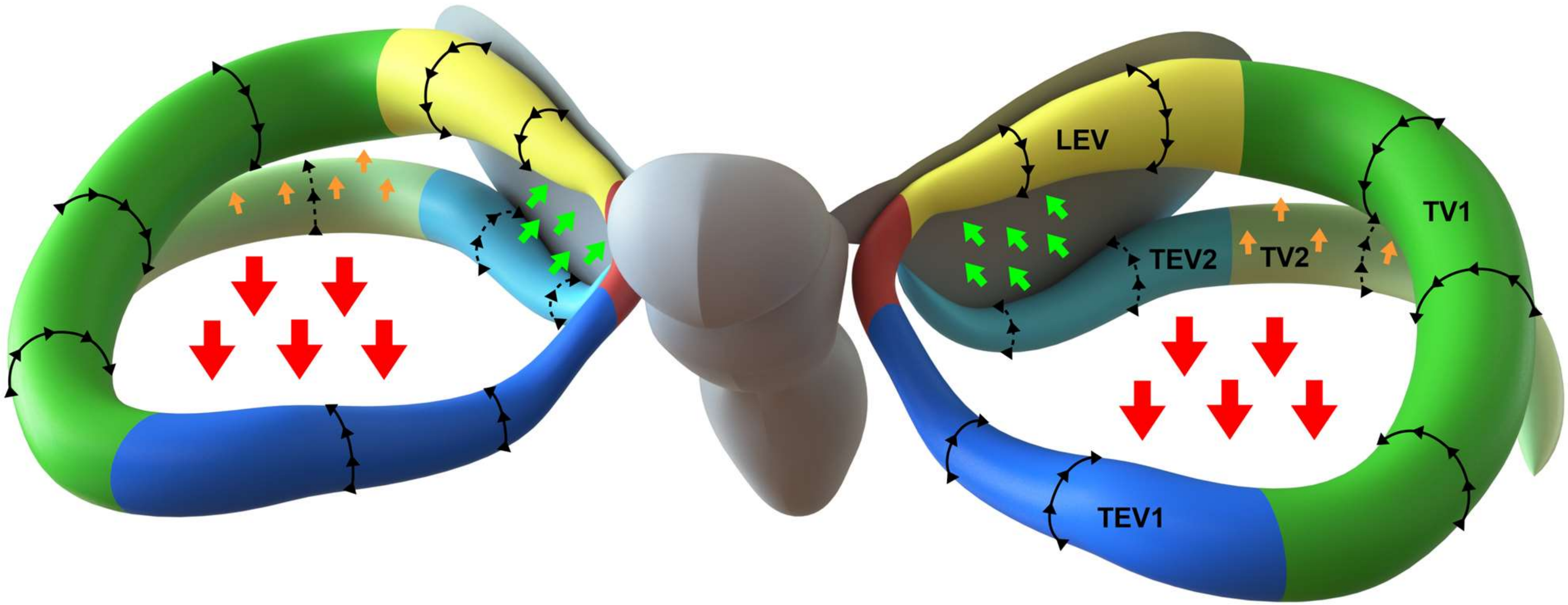


Flapping wing application

- **Flapping wing of insect type**
- **large amplitudes of wing motion + complicated kinematics (..with high frequencies)**
- **very high accelerations $\Rightarrow$ considerable 'added masses' effect**
- **highly unsteady flow (Strouhal number $St \sim 1$) / .... vortices at both edges !!**
- **low Reynolds number ($Re = O\,(10^2 \sim 10^0)$) / .... but very thin boundary layer !!**



Flapping wing: vortices and unsteady effects

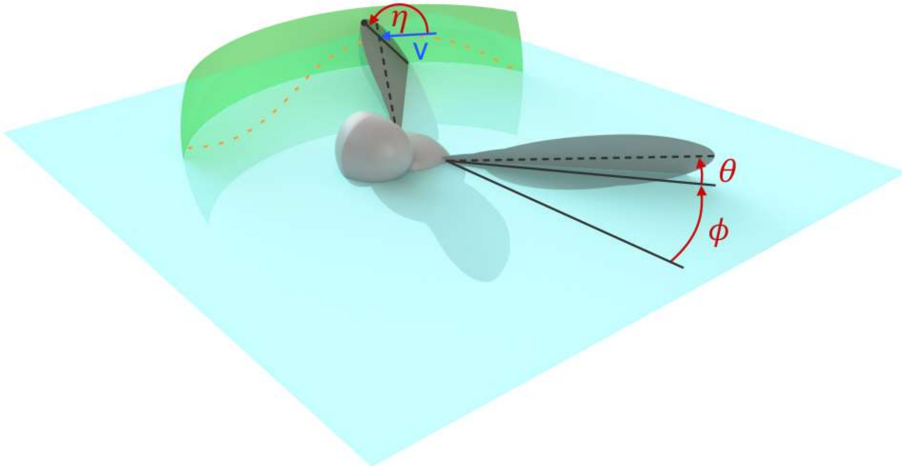


- Ellington, C.P., van den Berg, C.,Willmott, A.P., Thomas, A.L.R.: *Leading-edge vortices in insect flight*. Nature 384, 626-630 (1996)
- Karásek, M., Muijres, F.T., Wagter, C.D., Remes, B.D.W., de Croon, G.C.H.E. *A tailless aerial robotic flapper reveals that flies use torque coupling in rapid banked turns*. Science 361, 1089–1094 (2018)

- **LEV – Leading edge vortex**
- **TEV1 – Starting trailing edge vortex**
- **TEV2 – Stopping trailing edge vortex**
- **TV1 – Upper tip vortex**
- **TV2 – Lower tip vortex**

Flapping wing: quasi-steady aerodynamic model

: Lift enhancement due to the stabilized LEV during translation
: Reduction of the effective angle of attack due to the 'downwash'
: Lift enhancement due to the rapid rotation at the end of 'translation'
: 'Added mass effect'



Lat. Drosophila melanogaster

Sane, S.P., Dickinson, M.H.: *The aerodynamic effects of wing rotation and a revised quasi-steady model of flapping flight*. Journal of Experimental Biology 205(8), 1087–1096 (2002)

Translational forces

$$F_{TL} = \int_0^R \frac{1}{2}\rho r^2 \dot{\phi}^2 c(r) C_{TL}(\alpha) dr$$

$$F_{TD} = \int_0^R \frac{1}{2}\rho r^2 \dot{\phi}^2 c(r) C_{TD}(\alpha) dr$$

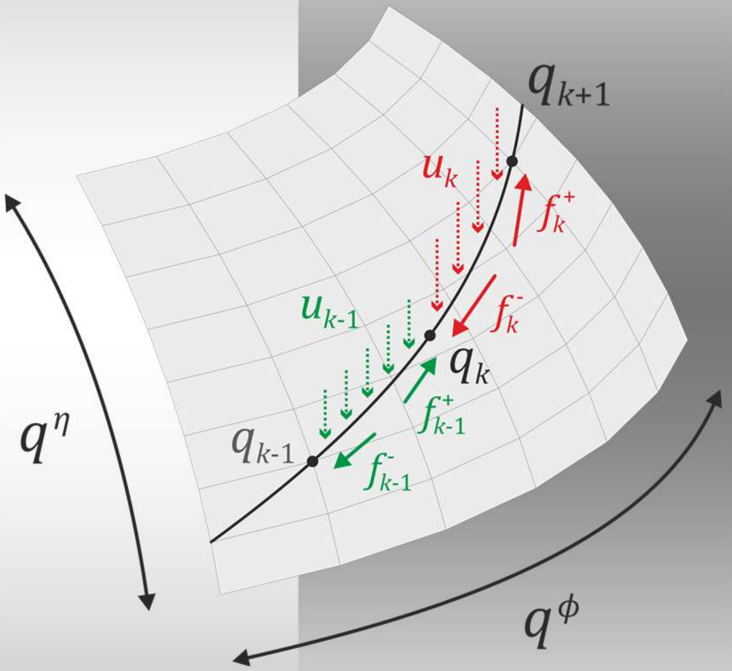
Rotational force

$$F_R = \rho C_R R^2 \dot{\phi}\dot{\eta}\overline{c}^2 \int_0^1 \hat{r}\hat{c}^2(\hat{r}) dr$$

Added mass force

$$F_A = \frac{\pi}{4}\rho R^2 \overline{c}^2 (\ddot{\phi}\sin\alpha + \dot{\phi}\dot{\alpha}\cos\alpha)\int_0^1 \hat{r}\hat{c}^2(\hat{r}) dr - \frac{\pi}{16}\rho\ddot{\alpha}R\overline{c}^3 \int_0^1 \hat{c}^2(\hat{r}) dr$$

DMOC optimisation procedure



$$\min_{q_d, u_d} J_d(q_d, u_d) = \sum_{k=0}^{N-1} C(q_k, q_{k+1}, u_k)$$

$$q_0 = q^0$$

$$q_N = q^T$$

$$p^0 + D_1 L_d(q_0, q_1) + f_0^- = 0$$

$$-p^T + D_2 L_d(q_{N-1}, q_N) + f_{N-1}^+ = 0$$

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_{k-1}^+ + f_k^- = 0$$

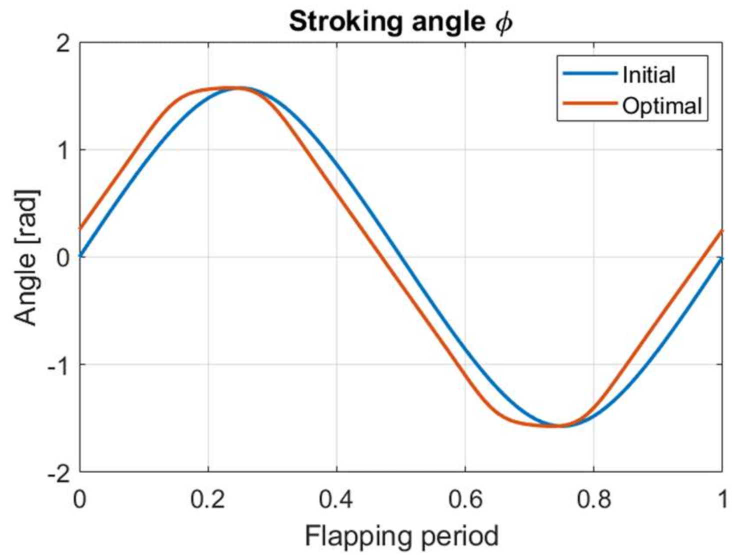
$$h_d(q_k, q_{k+1}, u_k) \geq 0$$

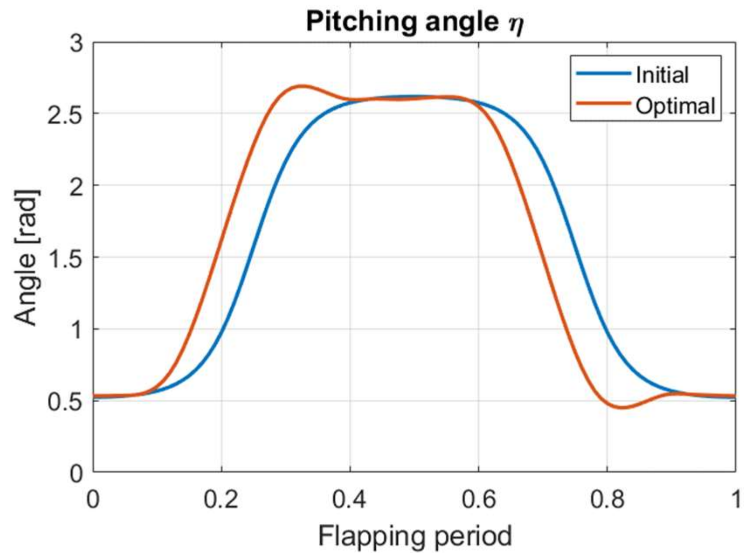
- Marsden, J.E., West, M.: *Discrete mechanics and variational integrators.* Acta Numerica 10, 357–514 (2001)
- Ober-Blöbaum, S., Junge, O., Marsden, J.E.: *Discrete mechanics and optimal control: An analysis.* ESAIM: Control, Optimisation and Calculus of Variations 17(2), 322–352 (2011)

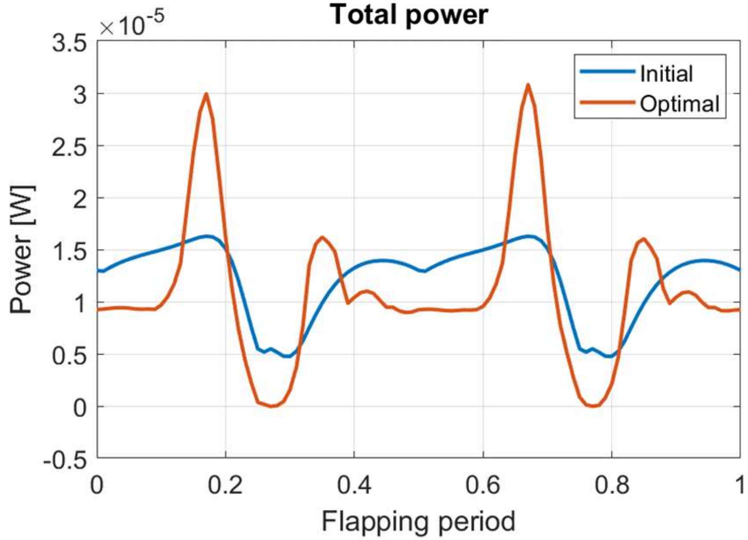
DEPARTMENT OF AERONAUTICAL ENGINEERING

CHAIR OF FLIGHT VEHICLE DYNAMICS

Flapping wing: optimised flapping pattern







Mean initial power $\rightarrow 1.2144 \times 10^{-5}$ W

Mean optimal power $\rightarrow 1.0907 \times 10^{-5}$ W

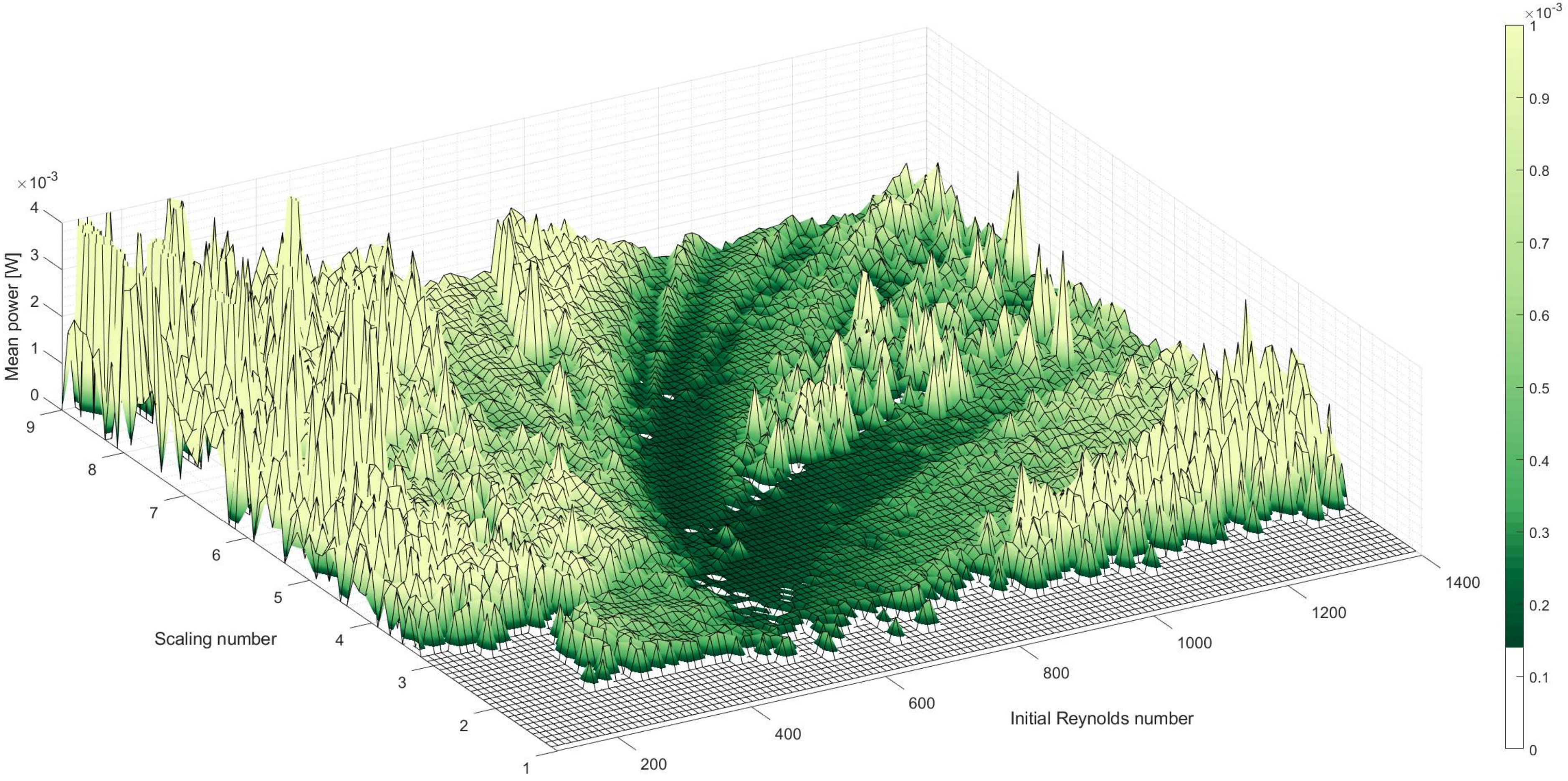
Improvement $\rightarrow 10.2\%$

Flapping wing on Mars

	Earth	Mars
Atmospheric density $[\mathrm{kg/m^3}]$	1.225	1.55×10^{-3}
Gravitational acceleration $[\mathrm{m/s^2}]$	9.81	3.72
Dynamic viscosity $[\mathrm{kg/ms}]$	1.8×10^{-5}	1.5×10^{-5}
Speed of sound $[m/\mathrm{s}]$	343	247

- Mars aircraft fly at low Re numbers due to a significant difference in atmospheric density and only a slight difference in gravitational acceleration, dynamic viscosity and speed of sound.

Flapping wing on Mars: optimisation of hover



- **Thank you for your**

  ATTENTION