# SSD Jean-Marie Souriau's book 50th birthday

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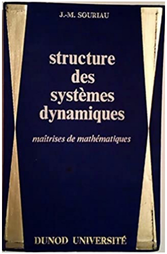
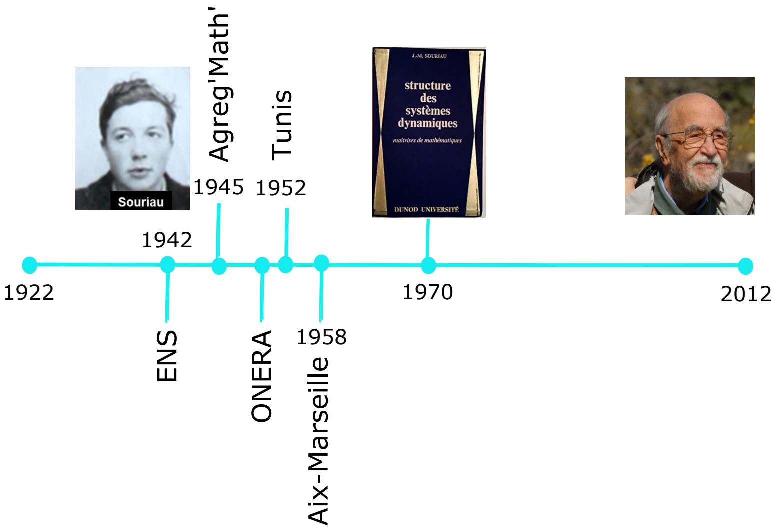
SPIG 2020 Les Houches

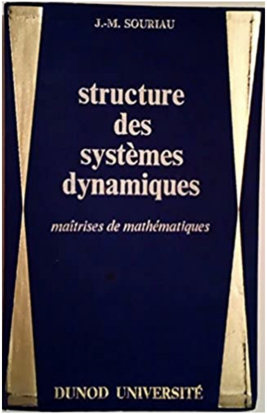


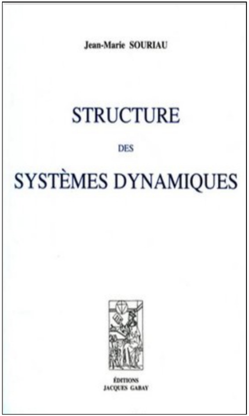


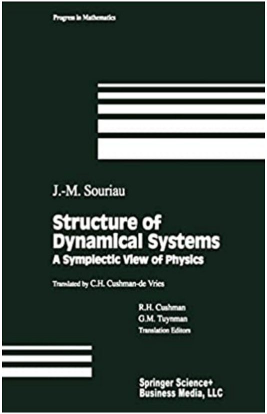

Société Française de Mathématiques

Gazette 133, juillet 2012

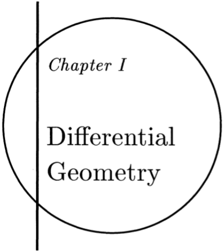

pourquoi mets-tu des lunettes de soleil pour lire Structure des Systèmes Dynamiques, de Souriau ?
parce que c'est d'une simplicité aveuglante
Structure des



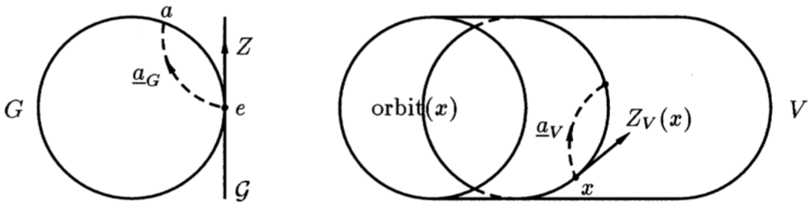




Le site officiel de Jean-Marie Souriau




- Manifolds
- Derivations
- Differential equations
- Differential forms
- Foliated manifolds
- Lie groups
- The calculus of variations



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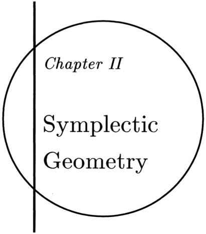
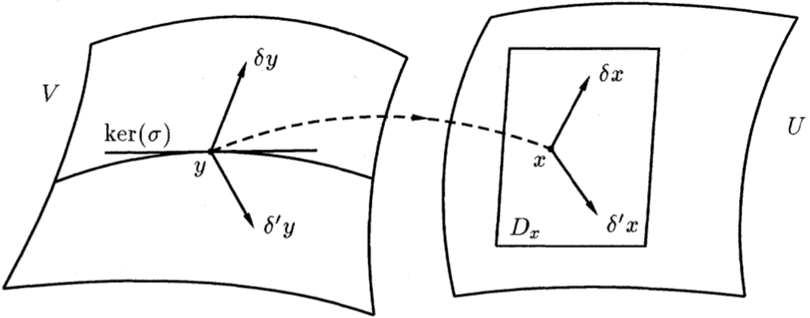



- 2-forms
- Symplectic manifolds
- Canonical transformations
- Dynamical groups

# Set of Leaves

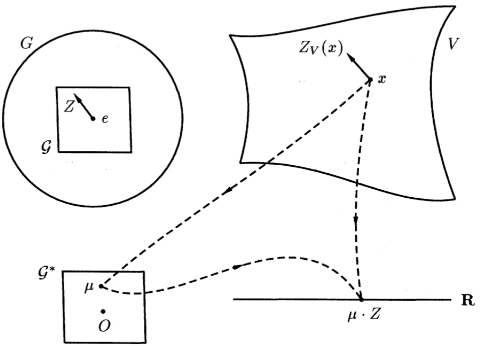



Thus there exists a vector, which we will call the *symplectic gradient* of $u$ and which we will denote by $\operatorname{grad} u$,[157] such that

$$-\,\mathrm{d}u \equiv \sigma(\operatorname{grad} u)\,.$$

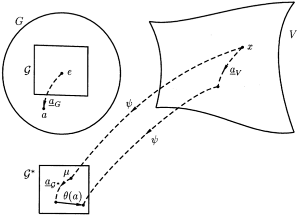
# Moment

$$\sigma(Z_V(x)) \equiv -\mathrm{d}[\mu \cdot Z] \qquad \text{for every constant } Z \in \mathcal{G}\,.^{173}$$




NOETHER'S THEOREM: Let $V$ be a *presymplectic* manifold and let $\mu$ be a moment of a dynamical group of $V$. Then $\mu$ is constant on each leaf of the characteristic foliation of $V$.

# Symplectic Cohomology of a dynamical group




(11.17) THEOREM: (See Fig. 11.IV.) Let $V$ be a *connected* symplectic (or presymplectic) manifold and let $G$ be a dynamical group of $V$ possessing a moment $\mu$ (11.7). Finally let $\psi$ denote the map $x \mapsto \mu$ from $V$ to the space $\mathcal{G}^*$ of torsors of $G$. Then

a) There exists a differentiable map $\theta$ from $G$ to $\mathcal{G}^*$ defined by [177]

$$\diamondsuit \qquad \theta(a) \equiv \psi(\underline{a}_V(x)) - \underline{a}_{\mathcal{G}^*}(\psi(x))\,.$$

b) The map $\theta$ satisfies the condition

$$\heartsuit \qquad \theta(a \times b) \equiv \theta(a) + \underline{a}_{\mathcal{G}^*}(\theta(b))\,.$$

c) The derivative $f = D(\theta)(e)$, where $e$ is the identity element of $G$, is a 2-form on the Lie algebra $\mathcal{G}$ of $G$ which satisfies

$$\clubsuit \qquad f(Z)([Z', Z'']) + f(Z')([Z'', Z]) + f(Z'')([Z, Z']) \equiv 0\,.$$

# Kirillov-Kostant-Souriau Theorem

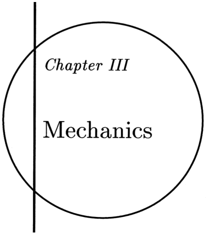
(11.34) THEOREM: Let $G$ be a Lie group, $\mathcal{G}$ its Lie algebra, and $\theta$ a symplectic cocycle of $G$. Furthermore, let $U$ be an orbit of the action $a \mapsto \underline{a}_{\mathcal{G}^*_\theta}$ (notation (11.28)) and let $\mu$ be a variable point in $U$. Then $U$ is a *submanifold* of $\mathcal{G}^*$, the space of torsors of $G$. A vector $\delta\mu$ is tangent to $U$ at $\mu$ if there exists a $Z \in \mathcal{G}$ such that

$$\heartsuit \qquad \delta\mu = \mu \cdot \mathrm{ad}(Z) + f(Z) \qquad (f = D(\theta)(e))\,.$$

Moreover, the dimension of $U$ (assumed to be nonzero) is *even* and $U$ admits the structure of a *symplectic manifold* whose Lagrange form $\sigma_U$ is given by

$$\diamondsuit \qquad \sigma_U(\delta'\mu)(\delta\mu) = \delta'\mu(Z) \qquad \text{if } \delta\mu = \mu \cdot \mathrm{ad}(Z) + f(Z)\,.$$

Finally, $G$, acting on $U$, is a dynamical group and each point $\mu \in U$ is its own moment.†




- The geometric structure of classical mechanics
- The principles of symplectic mechanics
- A mechanistic description of elementary particles
- Particle dynamics

## The Lagrange 2-form

Let us return to the evolution space $V$ and let us define a priori

$$\sigma(\delta y)(\delta' y) = \sum_j \Big( \langle m_j\,\delta \mathbf{v}_j - \mathbf{F}_j\,\delta t, \delta' \mathbf{r}_j - \mathbf{v}_j\,\delta' t \rangle - \langle m_j\,\delta' \mathbf{v}_j - \mathbf{F}_j\,\delta' t, \delta \mathbf{r}_j - \mathbf{v}_j \delta t \rangle \Big)\,.$$

which shows that the equation $\sigma(\delta y)(\delta y') = 0$ $[\forall\, \delta' y]$ can be written as

$$\begin{cases} m_j\,\delta \mathbf{v}_j - \mathbf{F}_j\,\delta t = 0 \\ \delta \mathbf{r}_j - \mathbf{v}_j\,\delta t = 0 \end{cases} \qquad \forall j\,.$$

It follows that *the equations of motion can be written as*

$$\sigma(\delta y) = 0$$

and that *the vector space* $\mathcal{E}$ *of* (12.27) *equals* $\ker(\sigma)$.

# Maxwell Principle

$$\mathbf{E}_j \equiv \mathbf{F}_j + \mathbf{B}_j \times \mathbf{v}_j$$

MAXWELL'S PRINCIPLE:[208] The Lagrange form $\sigma$ of a dynamical system has *zero exterior derivative* on the evolution space: $\mathrm{d}\sigma \equiv 0$.

If we substitute definition (12.45) of the form $\sigma$ into definition (4.32) of the exterior derivative and expand it, then after some computations, we obtain

$$\frac{\partial \mathbf{E}_j}{\partial \mathbf{v}_k} \equiv 0 \qquad \frac{\partial \mathbf{B}_j}{\partial \mathbf{v}_k} \equiv 0 \qquad \forall\, j, k$$

$$\overline{\frac{\partial \mathbf{E}_k}{\partial \mathbf{r}_j}} - \frac{\partial \mathbf{E}_j}{\partial \mathbf{r}_k} \equiv 0 \qquad \frac{\partial \mathbf{B}_j}{\partial \mathbf{r}_k} \equiv 0 \qquad \forall\, j \neq k$$

$$\operatorname{curl} \mathbf{E}_k + \frac{\partial \mathbf{B}_k}{\partial t} \equiv 0 \qquad \operatorname{div} \mathbf{B}_k \equiv 0 \qquad \forall\, k\,.$$ [209]

# Maxwell Principle

EXAMPLE: The $N$-body problem (12.8), given in an inertial frame by

$$\mathbf{B}_j \equiv 0\,, \qquad \mathbf{E}_j \equiv C \sum_{\substack{k \\ [k \neq j]}} m_j\, m_k \frac{\mathbf{r}_k - \mathbf{r}_j}{\|\mathbf{r}_k - \mathbf{r}_j\|^3}\,. \qquad \square$$

EXAMPLE: A *mass point* in a vacuum under the influence of *gravity*. In a reference frame fixed to the earth this is described by

$$\mathbf{E} \equiv m\,\mathbf{g}\,, \qquad \mathbf{B} \equiv 2m\,\mathbf{\Omega}\,,$$

where $\mathbf{g}$ is the *acceleration due to gravity* and $\mathbf{\Omega}$ is the *rotation vector of the earth.* $\square$

EXAMPLE: *Charged particles in an exterior electromagnetic field* for which we take

$$\mathbf{E}_j \equiv q_j\, \mathbf{E}(t, \mathbf{r}_j) + \sum_{\substack{k \\ [k \neq j]}} q_j\, q_k \frac{\mathbf{r}_j - \mathbf{r}_k}{\|\mathbf{r}_j - \mathbf{r}_k\|^3}$$

$$\mathbf{B}_j \equiv q_j\, \mathbf{B}(t, \mathbf{r}_j)\,.$$

# Space of Motions

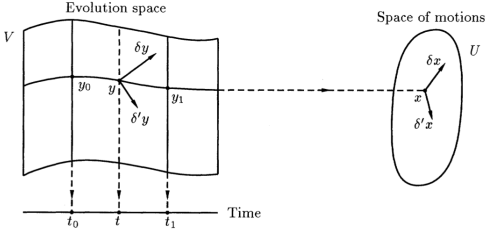
THEOREM: Let $V$ be the evolution space of a dynamical system satisfying Maxwell's principle, eventually supplemented with ideal holonomic constraints. Then

a) The Lagrange form $\sigma$ gives $V$ the structure of a *presymplectic manifold*.

b) Let $x$ denote the motion of the system defined by an initial condition $y$ (Fig. 12.II). Then the map $y \mapsto x$ is *differentiable*. On the space of motions $U$ there exists a 2-form, which we shall also call the *Lagrange form* and denote by $\sigma$, defined by

$$\diamondsuit \qquad \sigma(\delta y)(\delta' y) \equiv \sigma(\delta x)(\delta' x)\,.$$

This 2-form gives the space of motions the structure of a *symplectic manifold*. □



# Galilei group

Let us denote by $G$ the set of matrices $a$ considered in (12.71), namely

$$a \equiv \begin{bmatrix} A & \mathbf{b} & \mathbf{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{ll} A \in \mathrm{SO}(3)\,, & \mathbf{b} \in \mathbf{R}^3\,, \\ \mathbf{c} \in \mathbf{R}^3\,, & e \in \mathbf{R}\,. \end{array}$$

It is easy to verify that these matrices form a Lie group which is homeomorphic to $\mathrm{SO}(3) \times \mathbf{R}^7$ (and thus is connected and of dimension 10). This group is called the *Galilei group*. Its Lie algebra $\mathcal{G}$ is the set of matrices

$$Z \equiv \begin{bmatrix} j(\boldsymbol{\omega}) & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{ll} \boldsymbol{\omega} \in \mathbf{R}^3\,, & \boldsymbol{\beta} \in \mathbf{R}^3 \\ \boldsymbol{\gamma} \in \mathbf{R}^3\,, & \varepsilon \in \mathbf{R}\,. \end{array}$$

The Galilei group is a *Lie subgroup* (6.31) of the group of matrices $\mathrm{Gl}(\mathbf{R}^3 \times \mathbf{R}^2)$ (criterion (6.33) can be applied).

# Galilean moments

Since $\mu$ acts in a linear way on $\mathcal{G}$, we can write

$$\mu(Z) \equiv \langle \mathbf{l}, \boldsymbol{\omega} \rangle - \langle \mathbf{g}, \boldsymbol{\beta} \rangle + \langle \mathbf{p}, \boldsymbol{\gamma} \rangle + E\,\varepsilon\,,$$

$$\mathbf{l} \in \mathbf{R}^3\,,\ \mathbf{g} \in \mathbf{R}^3\,,\ \ \mathbf{p} \in \mathbf{R}^3\,,\ \ E \in \mathbf{R}\,;$$

we will denote the torsor $\mu$ defined this way by

$$\mu \equiv \{\mathbf{l},\, \mathbf{g},\, \mathbf{p},\, E\}\,.$$

EXAMPLE: Let us consider a material point of unit mass not subjected to any forces. A calculation gives immediately the following solution of (12.124)

$$\mu \equiv \{\mathbf{r} \times \mathbf{v},\, \mathbf{r} - \mathbf{v}\,t,\, \mathbf{v},\, \tfrac{1}{2}\,\|\mathbf{v}\|^2\}\,.$$

$$\theta_0(a) \equiv \psi(\underline{a}_V(y)) - \underline{a}_{\mathcal{G}^*}(\psi(y)) \qquad\qquad a \in G\,,\ y \in V\,.$$

A calculation gives

$$\theta_0(a) \equiv \{\mathbf{c} \times \mathbf{b},\, \mathbf{c} - \mathbf{b}\,e,\, \mathbf{b},\, \tfrac{1}{2}\,\|\mathbf{b}\|^2\}\,.$$

but straightforward calculation [215] shows that *the dimension of the symplectic cohomology space of the Galilei group is* 1. In other words, every symplectic cocycle $\theta$ is obtained from the cocycle $\theta_0$ (12.127) by the formula

$$\theta(a) \equiv \underline{a}_{\mathcal{G}^*}(\mu_0) - \mu_0 + m\,\theta_0(a)\,,$$

# Axioms of mechanics

I. The space of motions of a dynamical system is a *connected symplectic manifold*.

II. If several dynamical systems evolve independently, the manifold of motions of the composite system is the *symplectic direct product* of the spaces of motions of the component systems.

III. If a dynamical system is isolated, its manifold of motions admits the Galilei group as a dynamical group.

III. If a dynamical system is isolated, its manifold of motions admits the *restricted Poincaré group* as a dynamical group.

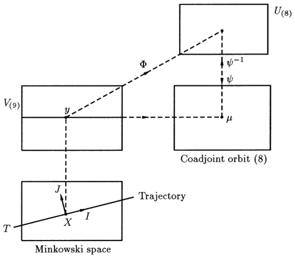
# Relativistic mechanics : Particle with Spin

$$W = *(M) \cdot P$$

The vector $W$ is called the *polarization*

DEFINITION: We will call an elementary dynamical system *a (relativistic) particle with spin* if its energy-momentum $P$ and polarization $W$ satisfy

$$\overline{P} \cdot P > 0 \qquad \text{and} \qquad W \neq 0\,.$$




THEOREM: For relativistic particles with spin we have the following collection of results.

a) $\overline{W} \cdot W$ is negative and the numbers

$$\diamondsuit \qquad m = \text{sign}(E)\sqrt{\overline{P} \cdot P}\ ^{244} \qquad \text{and} \qquad s = \sqrt{\frac{-\overline{W} \cdot W}{\overline{P} \cdot P}}$$

do not depend on the motion. They are called the *mass* [245] and *spin*

# Classification of Elementary Particles

### Case I. A particle with spin

DEFINITION: We will call an elementary dynamical system *a (relativistic) particle with spin* if its energy-momentum $P$ and polarization $W$ satisfy

$$\overline{P} \cdot P > 0 \qquad \text{and} \qquad W \neq 0\,.$$

### Case II. A particle without spin

DEFINITION: A relativistic *particle without spin* (or a *relativistic material point*) is an elementary dynamical system such that

$$\overline{P} \cdot P > 0 \qquad \text{and} \qquad W \equiv 0\,.$$

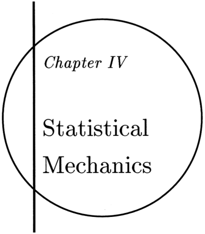
### Case III. A massless particle

DEFINITION: A *massless particle* [254] is an elementary dynamical system such that

$$\overline{P} \cdot P = \overline{W} \cdot W = 0$$

with both $P$ and $W$ nonzero.

### Nonrelativistic particles




- Measure on a manifold
- The principles of statistical mechanics

By a (generalized) *Gibbs measure* we will mean a probability measure $\zeta$ such that

$$\diamondsuit \quad \begin{cases} \exists\, z \in \mathbf{R}\ \exists\, Z \in E^* : \ \zeta = \lambda \times f \quad \text{with} \quad f(x) \equiv e^{-[z+Z(\Psi(x))]} \\ \Psi \text{ is } \zeta\text{-integrable.} \end{cases}$$

THEOREM: The $\lambda$-entropy of a Gibbs measure exists and is equal to

$$\heartsuit \qquad s = z + Z(M)\,,$$

## Equilibria of conservative systems

The "*natural*" equilibrium states of a system form the *Gibbs canonical ensemble* of the dynamical group of *time translations*.

A natural equilibrium state will thus be characterized by an element $Z$ of the Lie algebra of the Lie group $\mathbf{R}$, that is, $Z$ is a *real number*. We will see later on that $Z$ determines the *equilibrium temperature*.

## Covariant statistical mechanics

We propose the following principle.

If a dynamical system is invariant under a Lie subgroup $G'$ of the Galilei group, then the natural equilibria of the system form the Gibbs ensemble of the dynamical group $G'$.

A CENTRIFUGE ($\boldsymbol{\beta} = 0,\ \gamma = 0$).

With these assumptions we find

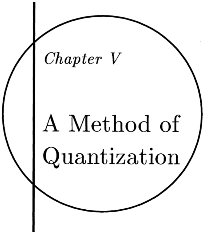
$$\mathbf{r} \equiv \exp(j(\boldsymbol{\omega}^* t))\, \mathbf{r}^* \,.$$

The new reference frame is thus *uniformly rotating*, where $\boldsymbol{\omega}^*$ is the *angular velocity vector*.[405]

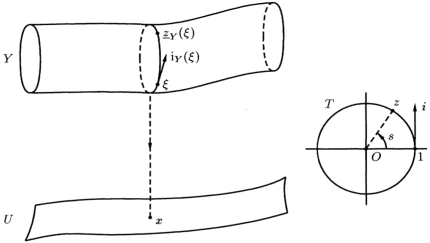
The probability of presence of the gas is proportional to

$$\exp\left(\frac{m}{2kT}\, \|\boldsymbol{\omega}^* \times \mathbf{r}^*\|^2\right).$$

The appearance of $m$ in the above expression shows — in the case of an inhomogeneous gas — that *the relative concentration of the various constituents varies with the distance to the axis of rotation*. This effect is well verified experimentally; it is used for the enrichment of uranium.




- Geometric quantization
- Quantization of dynamical systems


A Hausdorff manifold $Y$ will be called a *prequantum manifold* if

a) There exists a differentiable field of 1-forms $\xi \mapsto \varpi$ on $Y$ which defines a contact structure (18.2) on Y, that is,

$$\diamondsuit \qquad \dim(\ker\sigma)) \equiv 1 \qquad [\sigma \equiv \mathrm{d}\varpi]$$

$$\heartsuit \qquad \dim(\ker(\varpi) \cap \ker(\sigma)) \equiv 0\,.$$

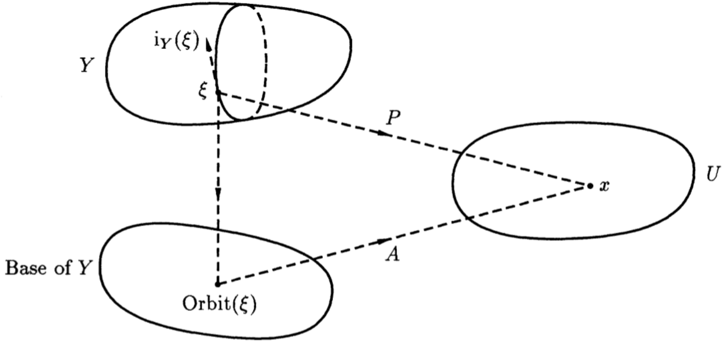
b) The torus $T$ *acts* on $Y$ (6.4) in such a way that [421]

$$\bowtie \qquad \underline{z}_Y(\xi) = \xi \iff z = 1 \qquad [z \in T]$$

$$\clubsuit \qquad \sigma(\mathrm{i}_Y(\xi)) \equiv 0$$

$$\spadesuit \qquad \varpi(\mathrm{i}_Y(\xi)) \equiv 1\,.$$

□

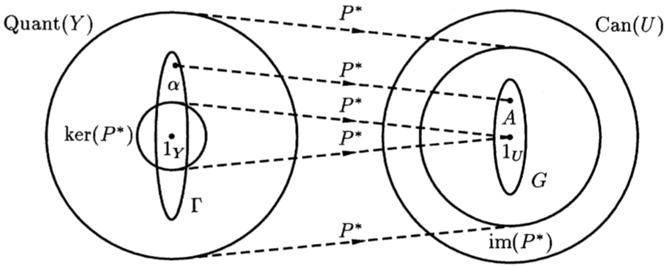



**Prequantization of a relativistic particle with spin $\frac{1}{2}$**

THEOREM: The relativistic particle with spin (model (14.4)) is prequantizable if and only if its spin satisfies

$$\diamond \qquad s = n\,\frac{\hbar}{2}\,, \qquad n \text{ an integer.}$$

# Quantomorphisms




If a dynamical group of a symplectic manifold is *quantizable, then its symplectic cohomology is zero.* □

We will see in (18.167) that this necessary condition *is not sufficient.*

In the case that a dynamical group $G$ is *liftable but not quantizable*, it might happen that one can find a Lie group $G'$ acting on $Y$ by quantomorphisms and providing a lift of $G$. Thus for $a \in G$ there would exist

*Thus there has to exist a classical system corresponding to every quantum mechanical system.*[461] If we assume this *correspondence principle*, it is legitimate to start with the classical description of a system in order to construct its quantum mechanical description. This is what one calls the *quantization* of the classical system.[462]

there exists a vector, which we will call the *symplectic gradient* of $u$ and which we will denote by $\operatorname{grad} u$,[157] such that

$$-\,\mathrm{d}u \equiv \sigma(\operatorname{grad} u)\,.$$

Let $(Y, P)$ be a prequantization of a symplectic manifold $U$. To every dynamical variable $u$ defined on $U$, we can associate an operator $\widehat{u}$ on $\mathcal{H}(Y)$ defined by[455]

$$\bowtie \qquad \widehat{u}(\Psi)(\xi) \equiv -i \underset{u}{\delta}[\Psi(\xi)] \qquad \forall\, \Psi \in \mathcal{H}(Y)\,.$$

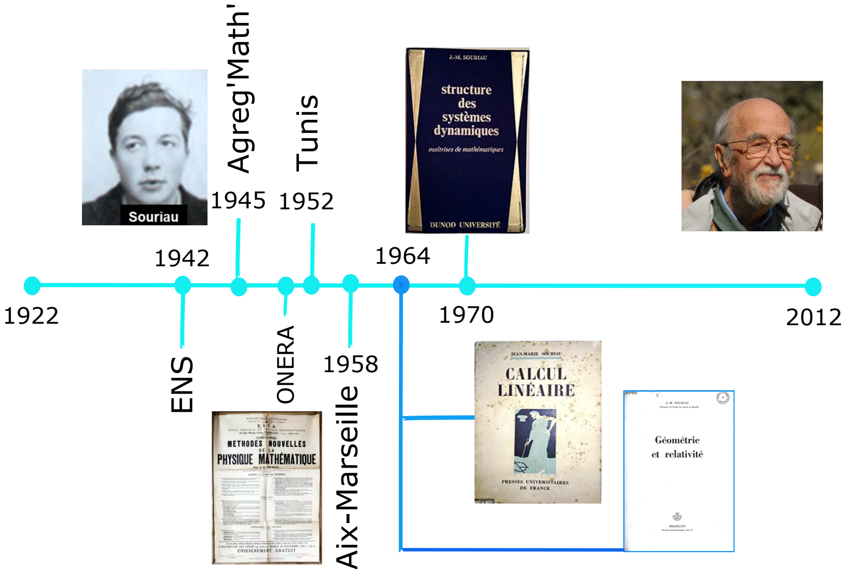
THEOREM:

♠ $\widehat{u}$ is a *hermitian* operator.

♡ The map $u \mapsto \widehat{u}$ is *linear* and *injective*.[456]

$$\diamondsuit \qquad \widehat{1} \;=\; 1_{\mathcal{H}(Y)}$$

$$\clubsuit \qquad \widehat{u} \circ \widehat{u'} - \widehat{u'} \circ \widehat{u} \;=\; -i\, \widehat{[u, u']_P}\,.$$ [457]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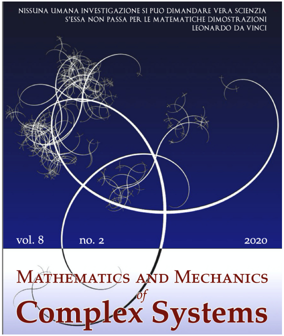
# Thank you !




Géry de Saxcé & Charles-Michel Marle

Presentation of Jean-Marie Souriau's book

"Structure des systèmes dynamiques"