Information Geometry and Quantum Fields

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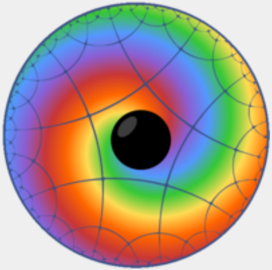


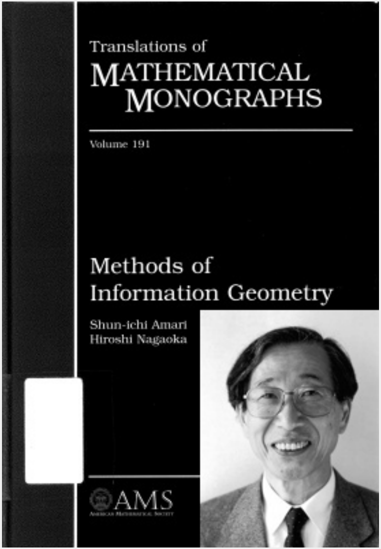
Complexity and Topology in Quantum Matter

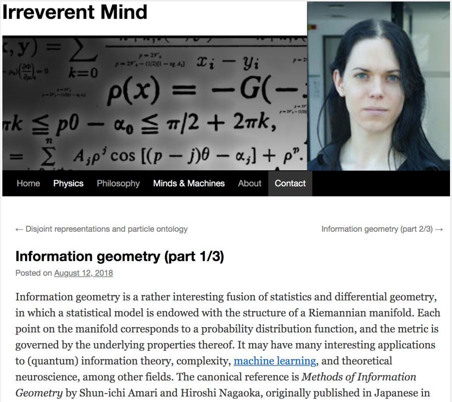
SciPost Phys. 8 (2020) 073 [arXiv:2001.02683]

Johanna Erdmenger

Ro Jefferson

Sources and References

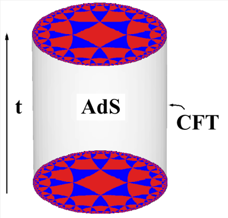




Outline

1. Introduction to AdS/CFT
2. Fisher metric, Symmetries and Uniqueness
3. Classical metric on States: Instantons
4. Classical metric on Theories: Ising Model
5. Quantum metric on States: Coherent states
6. Connections and Curvature
7. Conclusions and Outlook

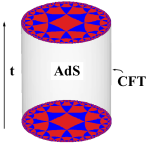
Holography: G. 't Hooft (THU-93/26, '93), L. Susskind (JMP **36**, '95)
AdS/CFT: J. Maldacena (ATMP **2**, '97)



Weak-strong duality.

$$\left\langle e^{\int d^dx\, \phi_0(x)\, \mathcal{O}(x)} \right\rangle_{\text{CFT}} = e^{-S_{\text{SUGRA}}}\Big|_{\phi(0,x)=\phi_0(x)}$$

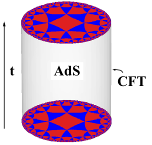
Anti-de Sitter Spacetime



Poincaré patch: $ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu$.

Constant negative curvature, maximally symmetric spacetime.

Vacuum of Einstein-Hilbert $S = \frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{-g}(R - 2\Lambda)$ with constant negative Λ (cosmological constant).



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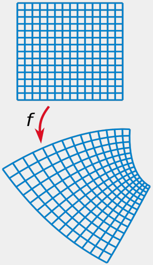
Embedding into $\mathbb{R}^{d,2}$:
$-(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -L^2$

Isometries: $\frac{d(d-1)}{2}$ rotations among $X^{1,\ldots,d}$, one rotation between X^0 and X^{d+1}, and $2d$ boosts mixing X^0 and X^{d+1} with $X^{1,\ldots,d}$.

Algebra of isometries is $\boxed{\mathfrak{so}(d,2)}$.

- Translations $P_\mu = i\partial_\mu$
- Rotations/boosts $J_{\mu\nu} = 2ix_{[\mu}\partial_{\nu]}$
- Dilatations $D = ix^\mu\partial_\mu$
- SCT $K_\mu = i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu)$
- Conformal algebra is $\mathfrak{so}(d,2)$.
- In $1+1$d, extended to infinite Virasoro algebra.



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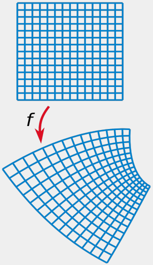
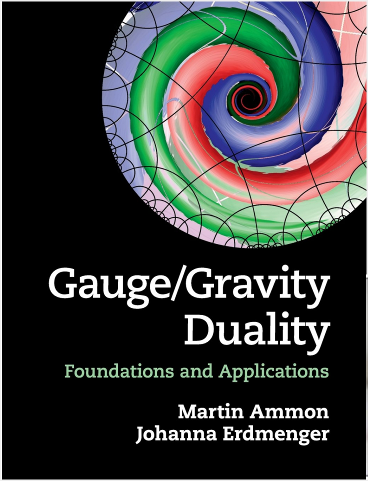


CFT fields form representations of the conformal algebra.


Gauge/Gravity Duality
Foundations and Applications
Martin Ammon
Johanna Erdmenger

- Entanglement probes of the bulk.

  [0603011 Ryu, Takayanagi; 0705.0016 Hubeny, Rangamani, Takayanagi; 1408.3203 Engelhardt, Wall]

- Quantum error correction $\implies$ bulk reconstruction.

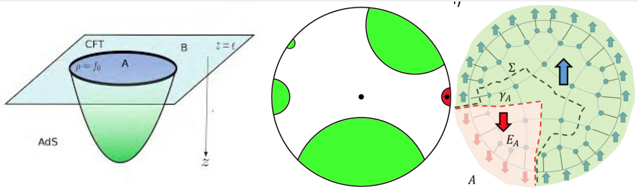
  [1411.7041 Almheiri, Dong, Harlow]

- Tensor networks $\implies$ bulk-boundary maps.

  [1601.01694 Hayden et. al.]

- Holographic distance measures.

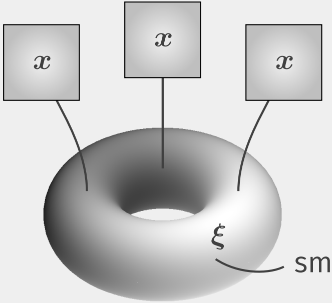
  [pure: 1507.0755 Lashkari, van Raamsdonk; mixed: 1701.02319 Banerjee, Erdmenger, Sarkar]



Previous Info Geom + Physics work

- Ruppeiner ('95,'96) - stat phys and thermo (e.g., ideal gas)
- Brody & Hook (2008) - stat phys and thermo (e.g., vdW gas)
- Janke et. al. (2002, 2003) - 1d Ising & Spherical models
- Dolan, Johnston, Kenna (2002) - Potts model
- Amari ('97) - Neural networks
- Ke & Nielsen (2016) - machine learning
- Heckman (2013) - string theory
- Blau, Narain, Thompson (2001) - YM instantons in 3+1-d

Fisher Metric



$\xrightarrow{\boldsymbol{p}} \mathbb{R}$

smooth, $\partial_i \equiv \frac{\partial}{\partial \xi^i}$

$$\boldsymbol{p_\xi(x) \geq 0}$$

$$\boldsymbol{\int dx\, p_\xi(x) = 1}$$

$$\boldsymbol{\xi} \xrightarrow{\text{injective}} \boldsymbol{p_\xi}$$

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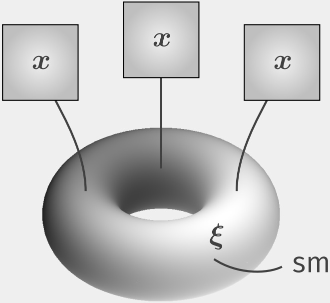

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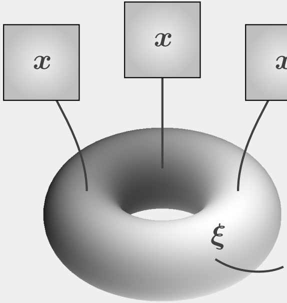

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Fact 1: $g_{ij}(\xi) = -\int dx\, p_\xi(x)\, \partial_i \partial_j \ln p_\xi(x)$ and $g_{ij}(\xi) = 4 \int dx\, \partial_i \sqrt{p_\xi(x)}\, \partial_j \sqrt{p_\xi(x)}$

Fact 2: g_{ij} is positive semi-definite.

A simple example (1d Gaussians):

Random variable space is $X = \mathbb{R}$.

Parameter space is $\Xi =$ upper half-plane $= (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_{>0}$

$$p_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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Comments:

1. It is *not* true that all free theories have flat Fisher metrics.
2. This hyperbolic space is a priori unrelated to AdS/CFT.

Generalization of the Gaussian (Exponential Family):

$p_\xi(x) = \exp\big[C(x) + \xi^i F_i(x) - \psi(\xi)\big]$.

$\psi(\xi) = \ln \int dx \, \exp\big[C(x) + \xi^i F_i(x)\big]$ is a normalization factor.

Assume that $\{C, F_i\}$ are lin. indep. so $\xi \to p_\xi$ is bijective.

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$\boxed{g_{ij}(\xi) = \partial_i \partial_j \psi(\xi)}$

Fact: $g_{ij}(\xi) = \langle (F_i - \langle F_i \rangle_\xi)(F_j - \langle F_j \rangle_\xi \rangle_\xi$ where $\langle f \rangle_\xi \equiv \int dx\, f(x)\, p_\xi(x)$.

Symmetries

$\xi \to \tilde{\xi}(\xi)$ is a *symmetry* of p if it can be *undone* by $x \to \tilde{x}(x)$

$$\boxed{p_{\tilde{\xi}}(x)\, dx = p_{\xi}(\tilde{x})\, d\tilde{x}}$$

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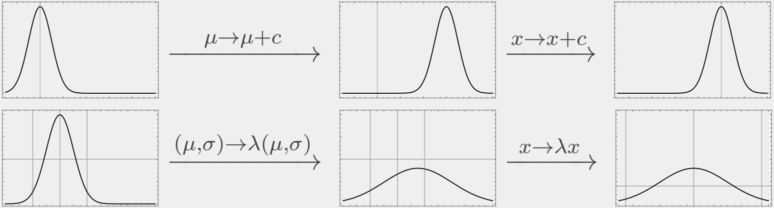
Gaussian example:



Non-Uniqueness

Infinitely many distributions give the same information geometry [Clingman, Murugan and Shock 2015]

E.g., 1d Gaussian and Cauchy-Lorentz give 2d hyperbolic space

Cauchy-Lorentz is *not* in the exponential family

Weirder example in Clingman, Murugan, Shock (2015) of a 3d distribution that gives 2d hyperbolic space, but has none of its symmetries.

- Many statistical models S can lead to the same Fisher metric (Gaussian and Cauchy-Lorentz).
- FM inherits the symmetries of S but can have more

Blau, Narain, Thompson (2001) - YM instantons in 3+1-d

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Simpler model: $\int d^4x \left(\partial_\mu\phi\,\partial^\mu\phi - g^2\phi^4\right)$ (Euclidean signature)

Exact solution to e.o.m.: $\phi_{\vec{\mu},\sigma}(\vec{x}) = \frac{2}{g\sigma}\,\frac{\sigma^2}{|\vec{x}-\vec{\mu}|^2+\sigma^2}$

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Lagrangian density: $\mathcal{L}_{\vec{\mu},\sigma}(\vec{x}) = \frac{16\sigma^2}{g^2} \frac{|\vec{x}-\vec{\mu}|^2-\sigma^2}{(|\vec{x}-\vec{\mu}|^2+\sigma^2)^4}$

$\mathcal{L}_{\vec{\mu},\sigma}(\vec{x}) < 0$ for $|\vec{x}-\vec{\mu}| < \sigma$, so not a good prob. distr.

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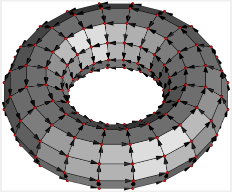
If instead we use $-\phi\, \partial^2 \phi - g^2 \phi^4$, we get $p_{\vec{\mu},\sigma}(\vec{x}) = \frac{16}{g^2} \frac{\sigma^4}{(|\vec{x}-\vec{\mu}|^2+\sigma^2)^4}$ and the information geometry is 5d hyperbolic space.

- Many statistical models S can lead to the same Fisher metric (Gaussian and Cauchy-Lorentz).
- FM inherits the symmetries of S but can have more
- FM on the states of a field theory can be sensitive to stability (scalar instanton).

Geometry of space of couplings of a theory.

2d classical Ising model on a square lattice:



$$H = -J \sum_{i,j=1}^{N} \sigma_{i,j}\sigma_{i+1,j} - K \sum_{i,j=1}^{N} \sigma_{i,j}\sigma_{i,j+1}$$

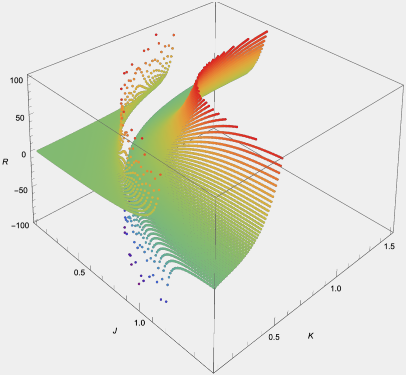
Partition function: $Z = \prod_i \sum_{\sigma_i = \pm 1} e^{-\beta H(\sigma)}$

Probability distribution: $p_{\beta J, \beta K}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}$.

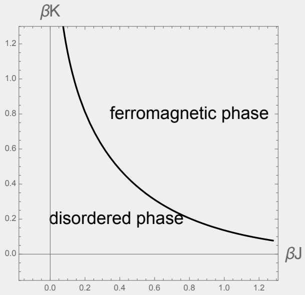
This is in the exponential family!

$g_{ij} = \partial_i \partial_j f$ where f is the reduced free energy per site

2D Ising Model Fisher Metric Ricci Scalar



(a) Ricci curvature as function of couplings J, K ($\beta = 1$)



(b) 2d Ising Model phase diagram for $\beta J, \beta K \geq 0$

The scalar curvature diverges at the critical curve given by

$$\sinh(2\beta J)\,\sinh(2\beta K) = 1$$

- Many statistical models S can lead to the same Fisher metric (Gaussian and Cauchy-Lorentz).
- FM inherits the symmetries of S but can have more
- FM on the states of a field theory can be sensitive to stability (scalar instanton).
- FM on the space of theories is sensitive to phase transitions (2d Ising model)

Set $J = K = 1$ and the critical temp. is $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right) \approx 0.44$.

$g_{\beta\beta}$ is the specific heat:

$$g_{\beta\beta} = \frac{d^2 f}{d\beta^2} \simeq \ln\frac{1}{|\beta - \beta_c|} \simeq \ln\frac{1}{|m|},$$

where $m = 2\left(\frac{\tanh\beta_c}{\tanh\beta} - 1\right)$.

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Well-known effective theory: $S = \int \frac{d^2 z}{2\pi}\left(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi} + im\overline{\psi}\psi\right)$.

Can we reproduce the $\ln\frac{1}{|m|}$ divergence of the Fisher metric in the effective field theory?

c.f. B. P. Dolan (1998): take p to be the *path integrand*.

In a QFT with action S, the Fisher metric on the couplings ξ^i is

$$g_{ij} = \frac{1}{\text{spacetime vol}} \Big(\langle \partial_i S \, \partial_j S \rangle - \langle \partial_i S \rangle \langle \partial_j S \rangle \Big).$$

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If ξ^i and ξ^j are masses, then $\langle \partial_i S\, \partial_j S\rangle$ is 4-pt and $\langle \partial_i S\rangle$ is 2-pt.

For a *free* theory, the 4-pt reduces to 2-pt fns.

For the free 1d Majorana fermion theory,

$$g_{mm} \simeq \int_0^{\Lambda} \frac{p\,dp}{p^2+m^2} \simeq \ln\frac{\Lambda}{|m|} \simeq \ln\frac{1}{|m|}$$

Quantum metric on States: Divergences

Divergence: bifunctional $D(p||q)$ of two distributions p and q.

D is a measure of how "different" q is from p ($D = 0$ iff $q = p$).

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e.g., $D^{(\alpha)}(p||q) = \frac{4}{1-\alpha^2}\left(1 - \int dx\, p^{\frac{1+\alpha}{2}} q^{\frac{1-\alpha}{2}}\right)$ for $\alpha \in \mathbb{R}$.

The $\alpha = 1$ limit is the Kullback-Leibler divergence, or relative entropy $D^{(1)}(p||q) = \int dx\, p \ln \frac{p}{q}$.

$$g_{ij}(\xi) = \frac{\partial^2}{\partial \xi'^i\, \partial \xi'^j} D(p_\xi||p_{\xi'})\Big|_{\xi'=\xi}$$

A *quantum state* is described by a *density matrix* ρ.

$p \to \rho$ and "$\int p\, dx = 1$" $\to$ "$\mathrm{tr}\rho = 1$".

Bures distance $D_B(\rho_1, \rho_2) = 2\big(1 - \big|\mathrm{tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\big|\big)$

For pure states $D_B(|\psi_1\rangle, |\psi_2\rangle) = 2\big(1 - \big|\langle\psi_1|\psi_2\rangle\big|\big)$.

The *Bures metric* is the leading (quadratic) term in the expansion of D_B around $\rho_2 \approx \rho_1$.

c.f. Nozaki, Ryu, Takayanagi (2012)

Two free Majorana fermions $\{a, a^\dagger\} = \{b, b^\dagger\} = 1$.

Fermionic coherent state: $|\psi_\lambda\rangle = \sqrt{\frac{1}{1+|\lambda|^2}}\, e^{-\lambda a^\dagger b^\dagger}|\Omega\rangle$ with $\lambda \in \mathbb{C}$.

One free complex scalar $[a, a^\dagger] = [b, b^\dagger] = 1$.

Bosonic coherent state: $|\psi_\lambda\rangle = \sqrt{1-|\lambda|^2}\, e^{-\lambda a^\dagger b^\dagger}|\Omega\rangle$ with $\lambda \in \mathbb{C}$.

Fermions: $ds^2 = \frac{d\lambda\, d\bar{\lambda}}{(1+|\lambda|^2)^2}$ and Bosons: $ds^2 = \frac{d\lambda\, d\bar{\lambda}}{(1-|\lambda|^2)^2}$

Fermions: 2-sphere and Bosons: 2d hyperbolic space

Symmetry of a family of density matrices: $\rho_\xi \to \rho_{\xi'} = U\rho_\xi U^\dagger$ for *some* (special) unitary matrix U.

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The Fermionic coherent state contains only $|\Omega\rangle$ and $a^\dagger b^\dagger|\Omega\rangle$.

In this 2d space, $\rho_\lambda = \frac{1}{2}\big(I + \hat{n}_\lambda \cdot \vec{\sigma}\big)$, where $\vec{\sigma} =$ Pauli matrices and $\hat{n}_\lambda$ is a real unit 3d vector built out of λ.

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A general 2×2 special unitary matrix takes the form $U = e^{i\theta \hat{n}\cdot\vec{\sigma}}$ for some angle θ and axis $\hat{n}$.

Conjugation of ρ_λ by $U = e^{i\theta \hat{n}\cdot\vec{\sigma}}$ rotates $\hat{n}_\lambda$ by θ around $\hat{n}$.

Symmetry algebra of ρ_λ is $\mathfrak{su}(2) = \mathfrak{so}(3)$.

The metric must be that of the 2-sphere.

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The Fermionic coherent state contains only $|\Omega\rangle$ and $a^\dagger b^\dagger |\Omega\rangle$.

In this 2d space, $\rho_\lambda = \frac{1}{2}\big(I + \hat{n}_\lambda \cdot \vec{\sigma}\big)$, where $\vec{\sigma} =$ Pauli matrices and $\hat{n}_\lambda$ is a real unit 3d vector built out of λ.

A general 2×2 special unitary matrix takes the form $U = e^{i\theta \hat{n} \cdot \vec{\sigma}}$ for some angle θ and axis $\hat{n}$.

Conjugation of ρ_λ by $U = e^{i\theta \hat{n} \cdot \vec{\sigma}}$ rotates $\hat{n}_\lambda$ by θ around $\hat{n}$.

Symmetry algebra of ρ_λ is $\mathfrak{su}(2) = \mathfrak{so}(3)$.

The metric must be that of the 2-sphere.

In fact, $ds^2 = \frac{1}{4} d\hat{n}_\lambda \cdot d\hat{n}_\lambda$.

The Bosonic coherent state has $\frac{1}{n!}(a^\dagger b^\dagger)^n|\Omega\rangle$ for $n = 0, 1, \ldots$.

The (m, n)-comp. of ρ_λ is $(\rho_\lambda)_{m,n} = (-1)^{m+n}(1 - |\lambda|^2)\lambda^m\overline{\lambda}^n$.

Symmetries of the Bures Metric

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The symmetry algebra of ρ_λ is $\mathfrak{so}(2,1)$.

$(1,2)$-rotation by θ: $U_{mn} = e^{im\theta}\delta_{mn}$.

$(1,3)$-boost by $\epsilon \ll 1$: $U_{mn} = \delta_{mn} + \frac{\epsilon}{2}(m\delta_{m,n+1} - n\delta_{m+1,n})$.

$(2,3)$-boost by $\epsilon \ll 1$: $U_{mn} = \delta_{mn} + \frac{i\epsilon}{2}(m\delta_{m,n+1} + n\delta_{m+1,n})$.

The metric must be that of the 2d hyperbolic space.

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In fact, $ds^2 = \frac{1}{4}d\hat{m}_\lambda^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} d\hat{m}_\lambda$ for some $\hat{m}_\lambda$.

- Many statistical models S can lead to the same Fisher metric (Gaussian and Cauchy-Lorentz).
- FM inherits the symmetries of S but can have more
- FM on the states of a field theory can be sensitive to stability (scalar instanton).
- FM on the space of theories is sensitive to phase transitions (2d Ising model)
- The Bures metric defines a metric on quantum states (coherent states).

α-connection: $\Gamma^{(\alpha)}_{ij,k} = \langle(\partial_i\partial_j\ell_\xi + \frac{1-\alpha}{2}\partial_i\ell_\xi\, \partial_j\ell_\xi)\partial_k\ell_\xi\rangle_\xi$ where $\ell = \ln p$.

Exercise: $\Gamma^{(\alpha)}_{ij,k} = \frac{\partial^3 D^{(\alpha)}(p_\xi||p_{\xi'})}{\partial\xi'^i\,\partial\xi'^j\,\partial\xi'^k}\Big|_{\xi'=\xi} - \partial_k g_{ij}$

$\alpha = 0$ corresponds to the standard Christoffel symbols.

$\alpha \neq 0$ is not a metric connection (ie., $\nabla^{(\alpha)}_k g_{ij} \neq 0$).

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$2+1$d Chern-Simons: $S = \frac{1}{4\pi}\mathrm{tr}\int\left(\Gamma\wedge d\Gamma + \frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma\right)$ gives the e.o.m. $d\Gamma + \Gamma\wedge\Gamma = 0$.

Vanishing 1-curvature could be a particular solution.

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- The Bures metric defines a metric on quantum states (coherent states).
- Many concepts of curvature. What do they all mean?

Outlook

- What do all the curvatures mean? What physics is captured?
- Can we apply these concepts to Nielsen's ideas of geometrizing the space of states to define complexity?
- Can we apply these concepts to holographic renormalization? Can we map RG flow as literal "loss of information"?
- How to get gravitational dynamics this way?
- Recent work by Iqbal and McGreevy associating a modified 3d Ising model with a string theory. Application here? Maybe following Heckman's info geom and string theory work?

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Thank you very much for listening!

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