Galilean Mechanics and Thermodynamics of Continua

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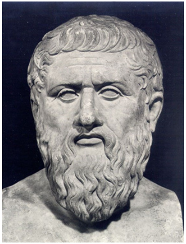
Idea debate

Preamble

$A\gamma\varepsilon\omega\mu\varepsilon\tau\rho\eta\tau o\varsigma\ \ \mu\eta\delta\varepsilon\iota\varsigma\ \ \varepsilon\iota\sigma\iota\tau\omega$

(Let none but geometers enter here !)

Plato



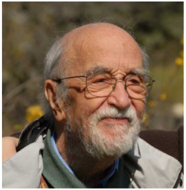
Idea debate

General Relativity

is not solely a theory of gravitation which would be reduced to predict tiny effects but –may be above all– it is a consistent framework for mechanics and physics of continua

Inspiration sources :

Jean-Marie Souriau
Lect. Notes in Math. 676 (1976)



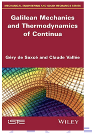
Claude Vallée, IJES (1981)

Some key ideas :

- We take the Relativity as model, process termed "geometrization", but with Galileo symmetry group
- The **entropy** is generalized in the form of a 4-vector and the **temperature** in the form of a 5-vector
- We generalize the **energy-momentum** tensor by associating the "mass" with it
- We decompose the new object into reversible and dissipative parts
- We obtain a covariant and more compact writing of the 1^{st} and 2^{nd} principles

Galilean Mechanics and Thermodynamics of Continua (2016)

Éléments de Mécanique galiléenne (2019)



Some key ideas :

- **Classical :** Clausius-Duhem inequality

$$\rho \, \frac{ds}{dt} - \frac{\rho}{\theta} \, \frac{dq_I}{dt} + div \, \left(\frac{h}{\theta} \right) \geq 0$$

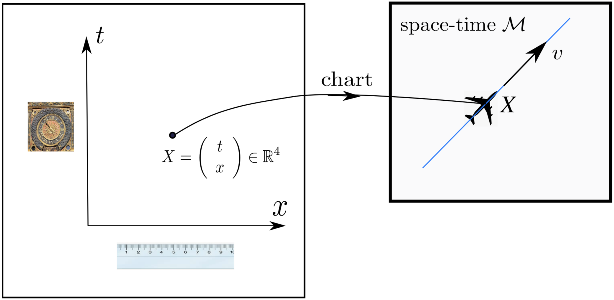
Truesdell (1952)

- **Relativistic :** 2nd principle Souriau (1976)

$$\boldsymbol{Div} \ \vec{\boldsymbol{S}} \geq 0$$

- Our aim is to find the classical counterpart of this principle of relativistic thermodynamics

Absolute space and time



Newton-Cartan structure :

- a 1-form $\boldsymbol{\tau} = dt$ (clock-form)
- a 2-contravariant symmetric tensor $\boldsymbol{h}$ of signature $(0 + ++)$

such that $\boldsymbol{\tau} \cdot \boldsymbol{h} = 0$

To know more : Duval, Küntzle, Trautman, Horváthy, Hartong, Bergshoeff, Van den Bleeken, ...

Galilean geometry

- **A geometry is a group action** Klein Erlangen program (1872)

  The **Galilean transformations** leave invariant :
  - the **durations** (or $\boldsymbol{\tau}$)
  - the **distances** (or $\boldsymbol{h}$)
  - **Uniform Straight Motion**

  then affine of the form $X = P\, X' + C$ with :

$$P = \begin{pmatrix} 1 & 0 \\ u & R \end{pmatrix}, \qquad C = \begin{pmatrix} \tau_0 \\ k \end{pmatrix}$$

  where $u \in \mathbb{R}^3$ is the **Galilean boost** and R is a rotation

- Their set is **Galileo's group**, a Lie group of dimension 10

The Galilean geometry is not Riemannian !

Galilean vectors

- Galilean vectors may be seen as **orbits** for the action of Galileo's group onto the vector components
- A Galilean vector $\vec{\boldsymbol{V}}$, represented by a column V, has a transformation law $V = P\, V'$ where P is a Galilean linear transformation
- The 4-velocity $\vec{\boldsymbol{U}}$ represented by the column

$$U = \frac{dX}{dt} = \begin{pmatrix} 1 \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix}$$

- Its transformation law $U = P\, U'$ provides the velocity addition formula

$$v = u + R\, v'$$

Galilean vectors

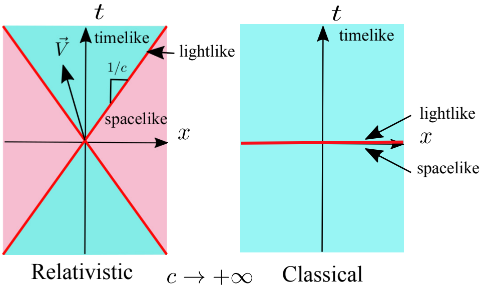
Theorem

A Galilean vector $\vec{\boldsymbol{V}}$ of non-vanishing time component V^0 is the 4-flux of it

For instance, the 4-flux of mass is $\vec{\boldsymbol{N}} = \rho\, \vec{\boldsymbol{U}}$

Conservation of V^0 is $Div\, \vec{\boldsymbol{V}} = 0$

Classification of Galilean vectors



The fifth dimension...

FIGURE – 5D simulator (La Foux, Saint-Tropez)

Bargmannian transformations

- We consider a line bundle $\pi : \mathcal{M} \to \hat{\mathcal{M}}$ of dimension 5 and a section :

$$\hat{f} : \mathcal{M} \to \hat{\mathcal{M}} \ : \ \boldsymbol{X} \mapsto \hat{\boldsymbol{X}} = \hat{f}\,(\boldsymbol{X})$$

- We built a group of affine transformations $\hat{X}' \mapsto \hat{X} = \hat{P}\,\hat{X}' + \hat{C}$ of $\mathbb{R}^5$ which are Galilean when acting onto the space-time hence of the form :

$$\hat{P} = \begin{pmatrix} P & 0 \\ \Phi & \alpha \end{pmatrix} \ ,$$

where P is Galilean, Φ and α must have a physical meaning linked to the energy

- Thus we know that, under the action of a boost u and a rotation R, the kinetic energy is transformed according to :

$$e = \frac{1}{2}\,m \parallel u + R\,v' \parallel^2 = \frac{1}{2}\,m \parallel u \parallel^2 + m\,u \cdot (R\,v') + \frac{1}{2}\,m \parallel v' \parallel^2 \ .$$

Bargmannian transformations

- We claim that the fifth dimension is linked to the energy by :

$$dz = \frac{e}{m}\, dt = \frac{1}{2} \parallel u \parallel^2 dt' + u^T R\, dx' + dz'$$

that leads to consider the **Bargmannian transformations** of $\mathbb{R}^5$ of which the linear part is :

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 \\ u & R & 0 \\ \frac{1}{2} \parallel u \parallel^2 & u^T R & 1 \end{pmatrix}$$

Their set is the **Bargmann's group**,
a Lie group of dimension 11,
introduced in quantum mechanics for cohomologic reasons
but which turns out very useful in Thermodynamics !

A fragance of symplectic geometry

- $(\mathcal{U}, \omega)$ symplectic manifold
  G Lie group acting on by $a \mapsto a \cdot \xi$
  $\mathfrak{g}^*$ the dual of its Lie algebra $\mathfrak{g}$ acting by $Z \mapsto Z \cdot \xi$
- $\xi \mapsto \mu = \psi(\xi) \in \mathfrak{g}^*$ is a **momentum map** (Souriau) if

$$\forall Z \in \mathfrak{g}, \qquad \omega(Z \cdot \xi, d\xi) = -d(\psi(\xi)\, Z)$$

- **Theorem** (Souriau)
  There exists $cocs : G \mapsto \mathfrak{g}^*$ called a **symplectic cocycle**
  such that $cocs(a) = \psi(a \cdot \xi) - Ad^*(a)\, \psi(\xi)$
- modulo a coboundary $cobs_{\mu_0} = Ad^*(a)\mu_0 - \mu_0$, it defines
  a **class of symplectic cohomology** $[cocs] \in H^1(G; \mathfrak{g}^*)$,
  generally null.
- A noticeable exception is **Galileo's group**

Bargman's group as central extension of Galileo's group

- The **extension** $\hat{G} = G \times N$ with N Abelian is a group for

$$\hat{a}\,\hat{a}' = (a,\theta)\left(a',\theta'\right) = \left(aa',\theta + \theta' + coc\left(a,a'\right)\right)$$

  if the N-**cocycle** *coc* verifies a cocycle identity ensuring the associativity
- one can define also a N-**coboundary** $cob_\theta\left(a,a'\right) = \theta\left(a\right) + \theta\left(a'\right) - \theta\left(aa'\right)$
- **Adjoint representation of** $\hat{G}$
  $Ad\left(\hat{a}^{-1}\right)(Z,Y) = \left(Ad\left(a^{-1}\right)Z, \quad Y + B(a)Z\right)$
  with $B(a) : \mathfrak{g} \to \mathfrak{n} : Z \mapsto Y = Dcoc_{(e,a)}\left(Z,0\right) + Dcoc_{(a^{-1},a)}\left(0,Za\right)$
- **Co-adjoint representation of** $\hat{G}$
  $Ad^*\left(\hat{a}\right)(\mu,\xi) = \left(Ad^*\left(a\right)\mu + C\left(a\right)\xi, \quad \xi\right)$
  with the transpose $C(a) : \mathfrak{n}^* \to \mathfrak{g}^*$ of $B(a)$

Bargman's group as central extension of Galileo's group

- reminder :
  $B(a)Z = Dcoc_{(e,a)}(Z,0) + Dcoc_{(a^{-1},a)}(0, Za), \quad C(a) = {}^t(B(a))$
- **Correspondance :**
  If *coc* is a N-cocycle, $a \mapsto C(a)\eta$ is a symplectic cocycle
  If *cob* is a N-coboundary, $a \mapsto C(a)\eta$ is a symplectic coboundary
- **Construction :** a group G with $[cocs] \neq 0$ being given, we find an extension $\hat{G}$ of null symplectic cohomology by determining the N-cocycle *coc* solution of the (non standard) PDS :

$$C(a)\,\xi = cocs\,(a)$$

- **Application :** G = Galileo's group, $\hat{G} = G \times \mathbb{R}$ = Bargmann's group

Temperature 5-vector

- The reciprocal temperature $\beta = 1 / \theta = 1 / k_B T$ is generalized as a Bargmannian 5-vector :

$$\hat{W} = \begin{pmatrix} W \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} ,$$

- The transformation law $\hat{W}' = \hat{P}^{-1} \hat{W}$ leads to :

$$\beta' = \beta \ , \quad w' = R^T (w - \beta u), \quad \zeta' = \zeta - w \cdot u + \frac{\beta}{2} \parallel u \parallel^2$$

- Picking up $u = w / \beta$, we obtain the **reduced form**

$$\hat{W}' = \begin{pmatrix} \beta \\ 0 \\ \zeta_{int} \end{pmatrix}$$

interpreted as the temperature vector of a volume element **at rest**

Temperature 5-vector

"Reduce and boost" method :
Starting from the reduced form, we apply the Galilean transformation of boost v, that gives :

$$\hat{W} = \begin{pmatrix} \beta \\ w \\ \zeta \end{pmatrix} , = \begin{pmatrix} \beta \\ \beta\, v \\ \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 \end{pmatrix} .$$

where ζ is **Planck's potential** or **Massieu's potential**

Friction tensor

Friction tensor

The **friction tensor** is a mixed 1-covariant and 1-contravariant tensor :

$$\boldsymbol{f} = \nabla \vec{\boldsymbol{W}}$$

represented by the 4×4 matrix $f = \nabla W$

- This object introduced by Souriau merges the temperature gradient and the strain velocity
- In dimension 5, we can also introduce

$$\hat{\boldsymbol{f}} = \nabla \hat{\vec{\boldsymbol{W}}}$$

represented by a 5×4 matrix

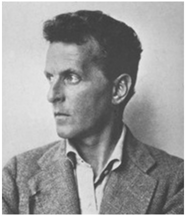
$$\hat{f} = \nabla \hat{W} = \begin{pmatrix} f \\ \nabla \zeta \end{pmatrix}$$

Momentum tensor

Method

Taking care **to walk up and down the rough ground of the reality** (Wittgenstein),

we want to work, in dimension 4 ou 5, with tensors of which the transformation law respects the physics



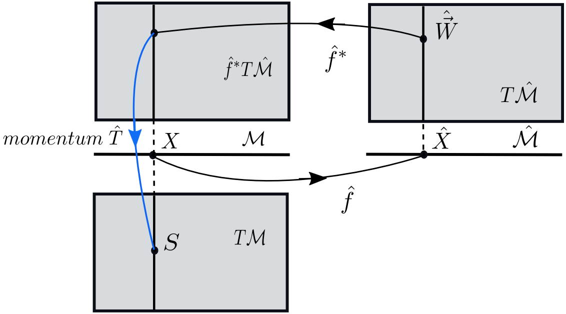
The meaning of the components is not given *a priori* but results, through the transformation law, from the choice of the symmetry group

Momentum tensor

Momentum tensor

Linear map from the tangent space to $\hat{\mathcal{M}}$ at $\hat{\boldsymbol{X}} = \hat{f}(\boldsymbol{X})$ into the tangent space to $\mathcal{M}$ at $\boldsymbol{X}$, hence a **mixed tensor** $\hat{\boldsymbol{T}}$ of rank 2

then a bundle map over the space-time $\hat{T} : \hat{f}^*(T\hat{\mathcal{M}}) \rightarrow T\mathcal{M}$



Momentum tensor

Momentum tensor

Linear map from the tangent space to $\hat{\mathcal{M}}$ at $\hat{\boldsymbol{X}} = \hat{f}(\boldsymbol{X})$ into the tangent space to $\mathcal{M}$ at $\boldsymbol{X}$, hence a **mixed tensor** $\hat{\boldsymbol{T}}$ of rank 2

- **Galilean momentum tensors :** represented by a 4×5 matrix of the form :

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ k & \sigma_\star & p \end{pmatrix}$$

where $\sigma_\star$ is a 3×3 symmetric matrix

- In matrix form, the transformation law is :

$$\hat{T}' = P\ \hat{T}\ \hat{P}^{-1}$$

To reveal the physical meaning of the components ...

Momentum tensor

... we let the symmetry group act !

- The transformation law provides :

$$\rho' = \rho \,, \quad p' = R^T\,(p - \rho u), \quad \sigma'_\star = R^T\,(\sigma_\star + u\,p^T + p\,u^T - \rho\,u\,u^T)\,R$$

$$\mathcal{H}' = \mathcal{H} - u \cdot p + \frac{\rho}{2} \parallel u \parallel^2, \quad k' = R^T\,(k - \mathcal{H}' u + \sigma_* u + \frac{1}{2} \parallel u \parallel^2 \,p)$$

- which leads to the **reduced form** :

$$\hat{T}' = \begin{pmatrix} \rho\, e_{int} & 0 & \rho \\ h' & \sigma' & 0 \end{pmatrix} ,$$

interpreted as the momentum of a volume element **at rest**

Momentum tensor

"Reduce and boost" method : starting from the reduced form, we apply a Galilean transformation law of boost v and rotation R and we interpret :

- ρ as the **density**
- $p = \rho\, v$ as the **linear momentum**
- $\sigma_\star = \sigma - \rho v v^T$ as the **dynamical stresses**
- $\mathcal{H} = \rho\, \left(e_{int} + \frac{1}{2} \parallel v \parallel^2\right)$ as the **total energy**
- $k = h + \mathcal{H} v - \sigma v$ as the **energy flux**

with :

- the **heat flux** $h = R\, h'$
- the **statical stresses** $\sigma = R\, \sigma'\, R^T$

Momentum tensor

Hence the **boost method** reveals the standard form of a

Galilean momentum tensor

Object structured into :

- **density** ρ,
- **linear momentum** p,
- **Cauchy's statical stresses** σ,
- **heat flux** h,
- **Hamiltonian** (per volume unit) $\mathcal{H}$

represented by the matrix :

$$\hat{T} = \begin{pmatrix} \mathcal{H} & -p^T & \rho \\ h + \mathcal{H}\,\frac{p}{\rho} - \sigma\,\frac{p}{\rho} & \sigma - \frac{1}{\rho}\,p\,p^T & p \end{pmatrix}$$

First principle

Momentum divergence

5-row $div\ \hat{T}$ such that, for all smooth 5-vector field $\hat{W}$:

$$Div\ (\hat{T}\ \hat{W}) = (Div\ \hat{T})\ \hat{W} + Tr\ \left(\hat{T}\ \nabla\ \hat{W}\right)$$

Covariant form of the 1^{st} principle

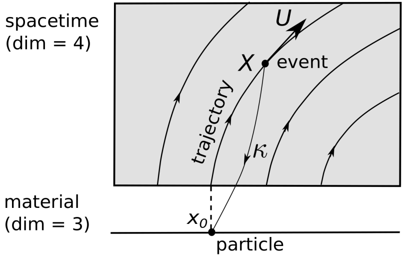
$$Div\ \hat{T} = 0$$

First principle

In absence of gravity, we recover the balance equations of :

- mass : $\dfrac{\partial \rho}{\partial t} + div\,(\rho\, v) = 0$
- linear momentum : $\rho \left[\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x}\, v \right] = (div\ \sigma)^T$,
- energy : $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \ (h + \mathcal{H} v - \sigma v) = 0$

The matter and its motion



Its motion is described by a **line bundle** $\kappa : \mathcal{M} \rightarrow \mathcal{M}_0 : \boldsymbol{X} \mapsto \boldsymbol{x}_0 = \kappa(\boldsymbol{X})$ where

- the particle $\boldsymbol{x}_0$ is represented by its Lagrangian coordinates x_0
- its trajectory is the fiber $\kappa^{-1}(\boldsymbol{x}_0)$
- the position x of the event $\boldsymbol{X}$ gives its Eulerian coordinates at time t
- the **deformation gradient** is $F = \frac{\partial x}{\partial x_0}$

First principle

Reversible medium

if Planck's potential ζ is a function of :
- the temperature vector W,
- and the right Cauchy strain $C = F^T F$,

then the 4×4 matrix

$$T_R = U\,\Pi_R + \begin{pmatrix} 0 & 0 \\ -\sigma_R v & \sigma_R \end{pmatrix}$$

with $\Pi_R = -\rho\,\frac{\partial \zeta}{\partial W}$ $\qquad \sigma_R = -\frac{2\rho}{\beta}\,F\,\frac{\partial \zeta}{\partial C}\,F^T$ is such that :

♡ $\hat{T}_R = \begin{pmatrix} T_R & N \end{pmatrix}$ with $N = \rho\,U$ represents a momentum tensor $\hat{\boldsymbol{T}}_R$

♢ $Tr\left(\hat{T}_R \nabla \hat{W}\right) = 0$

♣ $\hat{T}_R\;\hat{W} = \left(\zeta - \frac{\partial \zeta}{\partial W}\;W\right)\;N$

First principle

Planck's potential ζ is the prototype of **thermodynamic potentials** :

- the **internal energy** $e_{int} = -\dfrac{\partial \zeta_{int}}{\partial \beta}$
- the Galilean 4-vector $\vec{\boldsymbol{S}} = \hat{\boldsymbol{T}}_R \, \hat{\vec{\boldsymbol{W}}}$ is the 4-flux $\vec{\boldsymbol{S}} = s \, \vec{\boldsymbol{N}}$ of the **specific entropy** $s = \zeta_{int} - \beta \, \dfrac{\partial \zeta_{int}}{\partial \beta}$
- the **free energy** $\psi = -\dfrac{1}{\beta} \, \zeta_{int} = -\theta \, \zeta_{int}$ allows to recover

$$-e_{int} = \theta \, \frac{\partial \psi}{\partial \theta} - \psi, \qquad -s = \frac{\partial \psi}{\partial \theta}$$

The interest of ζ is that it generates all the other ones

Geometrization : $\mathcal{S} = \frac{\mathcal{Q}_R}{\theta} = \mathcal{Q}_R \cdot \beta$ becomes $\vec{\boldsymbol{S}} = \hat{\boldsymbol{T}}_R \, \hat{\vec{\boldsymbol{W}}}$

Second principle

Additive decomposition of the momentum tensor

$\hat{\mathrm{T}} = \hat{\mathrm{T}}_R + \hat{\mathrm{T}}_I$ with

- the reversible part $\hat{\mathrm{T}}_R$ represented by :

$$\hat{T}_R = \begin{pmatrix} \mathcal{H}_R & -p^T & \rho \\ \mathcal{H}_R v - \sigma_R v & \sigma_R - v p^T & \rho v \end{pmatrix}$$

- the irreversible one $\hat{\mathrm{T}}_I$ represented by :

$$\hat{T}_I = \begin{pmatrix} \mathcal{H}_I & 0 & 0 \\ h + \mathcal{H}_I v - \sigma_I v & \sigma_I & 0 \end{pmatrix}$$

where σ_I are the **dissipative stresses** and $\mathcal{H}_I = -\rho q_I$ is the dissipative part of the energy due to the **irreversible heat sources** q_I

Second principle

Clock-form

Linear form $\boldsymbol{\tau} = dt$ represented by an invariant row under Galilean transformation :

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Covariant form of the second principle

The **local production of entropy** of a medium caracterized by a temperature vector $\hat{\vec{\boldsymbol{W}}}$ and a momentum tensor $\hat{\mathrm{T}}$ is non negative :

$$\Phi = \boldsymbol{Div}\ \left(\hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}}\right) - \left(\boldsymbol{\tau}(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\ \left(\boldsymbol{\tau}(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0$$

and vanishes if and only if the process is reversible
[de Saxcé & Vallée IJES 2012]

Second principle

- The local production of entropy

$$\Phi = \boldsymbol{Div}\ \left(\hat{\boldsymbol{T}}\ \hat{\vec{\boldsymbol{W}}}\right) - \left(\tau(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\ \left(\tau(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right)$$

  is a **Galilean invariant !**

- After some manipulations, it can be putted in the classical form of **Clausius-Duhem inequality**

$$\Phi = \rho\ \frac{ds}{dt} - \frac{\rho}{\theta}\ \frac{dq_I}{dt} + div\ \left(\frac{h}{\theta}\right) \geq 0$$

Galilean coordinates

Theorem

A necessary and sufficient condition for the Jacobian matrix $P = \frac{\partial X'}{\partial X}$ of a coordinate change $X \mapsto X'$ being a linear Galilean transformation is that this change is compound of a rigid motion and a clock change :

$$x' = (R(t))^T (x - x_0(t)), \qquad t' = t + \tau_0$$

- The coordinate systems that are deduced one from each other by such changes are called **Galilean coordinate systems**.
- G being the group of linear Galilean transformations, this theorem shows that the G-**structure** [Kobayashi 1963] is **integrable**

Galilean gravitation

Theorem

The **Galilean connexions**, that is the symmetric connections of which the matrix Γ belongs to the Lie algebra of Galileo's group, are such that :

$$\Gamma(dX) = \begin{pmatrix} 0 & 0 \\ \Omega \times dx - g\, dt & j(\Omega)\, dt \end{pmatrix},$$

where $j(\Omega)$ is the unique skew-symmetric matrix such that $j(\Omega)v = \Omega \times v$

- g is the classical **gravity**
- Ω is a new object called **spinning**

Equation of motion of a particle

- $T = m\,U$ being the linear 4-momentum, the covariant equation of motion reads [Élie Cartan 1923] :

$$\nabla T = dT + \Gamma(dX)\,T = 0$$

or in tensor notations

$$\nabla T^{\alpha} = dT^{\alpha} + \Gamma^{\alpha}_{\mu\beta} dX^{\mu} T^{\beta} = 0$$

- In the Galilean coordinate systems, its general form is

$$\dot{m} = 0, \quad \dot{p} = m\,(g - 2\,\Omega \times v)$$

[Souriau, Structure des systèmes dynamiques, 1970]

- It allows to explain simply the motion of Foucault's pendulum without neglecting the centripetal force as in the classical textbook

Thermodynamics and Galilean gravitation

The down side of the cards ...

- Galileo's group does not preserve space-time metrics
- Bargmann's group preserves the metrics $ds^2 = \| dx \|^2 - 2\, dz\, dt$, then the space $\hat{\mathcal{M}}$ is a riemannian manifold and, in this case, the G-structure is not in general integrable, the obstruction being the curvature.
- In other words, we are going to work, up to now, in linear frames which are not associated to local coordinates (moving frames)
- Hence we have to find frames associated to coordinate systems (natural frames)

Thermodynamics and Galilean gravitation

- With the **potentials of the Galilean gravitation** ϕ, A such that

$$g = -grad\,\phi - \frac{\partial A}{\partial t}, \quad \Omega = \frac{1}{2}\,curl\,A$$

the Lagrangian is $\mathcal{L}(t, x, v) = \frac{1}{2}\,m \parallel v \parallel^2 - m\,\phi + m\,A \cdot v$

- that suggests to introduce a coordinate change

$$dz' = \frac{\mathcal{L}}{m}\,dt = dz - \phi\,dt + A \cdot dx\ , \quad dt' = dt, \quad dx' = dx$$

- In the new coordinates, the **Bargmannian connection** is

$$\hat{\Gamma}(d\hat{X}) = \begin{pmatrix} 0 & 0 & 0 \\ j(\Omega)\,dx - g\,dt & j(\Omega)\,dt & 0 \\ \begin{array}{c} \left(\dfrac{\partial \phi}{\partial t} - A \cdot g\right)\,dt \\ +\,(grad\,\phi - \Omega \times A) \cdot dx \end{array} & \begin{array}{c} [(grad\,\phi - \Omega \times A)\,\,dt \\ -grad_s A\,dx]^T \end{array} & 0 \end{pmatrix}$$

Thermodynamics and Galilean gravitation

The developments are similar to the ones in absence of gravitation but with some exceptions :

- Planck's potential becomes $\zeta = \zeta_{int} + \frac{\beta}{2} \parallel v \parallel^2 - \beta\, \phi + A \cdot w$
- the Hamiltonian becomes $\mathcal{H} = \rho\, \left(e_{int} + \frac{1}{2} \parallel v \parallel^2 + \phi - q_I \right)$,
- the linear momentum becomes $p = \rho\, (v + A)$.

In presence of gravitation, the first principle restitutes the balance equations of the mass and of

- the linear momentum : $\rho\, \dfrac{dv}{dt} = (div\ \sigma)^T + \rho\, (g - 2\, \Omega \times v)$
- the energy : $\dfrac{\partial \mathcal{H}}{\partial t} + div\ \ (h + \mathcal{H} v - \sigma v) = \rho \left(\dfrac{\partial \phi}{\partial t} - \dfrac{\partial A}{\partial t} \cdot v \right)$

A smidgen of relativistic Thermodynamics

- We come back to the relativistic model with **Lorentz-Poincaré** symmetry group

- In this approach, the temperature is transformed according to $\theta' = \frac{\theta}{\gamma} = \theta \sqrt{1 - \frac{\|v\|^2}{c^2}}$

  This the **temperature contraction** !

- thanks to Minkowski's space-time metrics $ds^2 = c^2 dt^2 - \parallel dx \parallel^2$, we can associate to the 4-velocity $\vec{\boldsymbol{U}}$ a single linear form $\boldsymbol{U}^*$ represented by

$$U^T G = \left(\gamma, \gamma\, v^T\right) \begin{pmatrix} c^2 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{pmatrix} = c^2 \left(\gamma, -\frac{1}{c^2}\gamma\, v^T\right) \ ,$$

which approaches $c^2 \boldsymbol{\tau}$ when c approaches $+\infty$

A smidgen of relativistic Thermodynamics

By an epistemological reversal, we replace the clock-form $\boldsymbol{\tau}$ by $\boldsymbol{U}^*/\ c^2$ in the Galilean expression of the 2^{nd} principle, that lead to

Relativistic form of the 2^{nd} principle

The **local production of entropy** of a medium characterized by a temperature vector $\vec{\mathrm{W}}$, a momentum tensor $\hat{\boldsymbol{T}}$, a potential ζ and a 4-flux of mass $\vec{\boldsymbol{N}}$ is non negative :

$$\Phi = \boldsymbol{Div}\ \left(\boldsymbol{T}\,\vec{\boldsymbol{W}} + \zeta\,\vec{\boldsymbol{N}}\right) - \frac{1}{c^2}\left(\boldsymbol{U}^*(\boldsymbol{f}(\vec{\boldsymbol{U}}))\right)\frac{1}{c^2}\left(\boldsymbol{U}^*(\boldsymbol{T}_I(\vec{\boldsymbol{U}}))\right) \geq 0\ ,$$

and vanishes if and only if the process is reversible

Thank you !

Lie group statistical mechanics

In `Structure des systèmes dynamiques (1970)`, Souriau proposed a statistical mechanics model using geometric tools

- Let $d\lambda$ be a measure on the orbit $orb\,(\mu)$, identified to $\boldsymbol{\mu}$, and a Gibbs probability measure $p\,d\lambda$ with $p = e^{-\Theta(\mu)} = e^{-(z+\mu\,Z)}$
- The normalization condition $\int_{orb(\mu)} p\,d\lambda = 1$ links the components by

$$z(Z) = \ln \int_{orb(\mu)} e^{-\mu\,Z}\,d\lambda$$

- The corresponding **entropy** and mean momenta are :

$$s = -\int_{orb(\mu)} p\,\ln p\,d\lambda = z + M\,Z, \qquad M = \int_{orb(\mu)} \mu\,p\,d\lambda = -\frac{\partial z}{\partial Z}$$

Bridging the gap between both theories :

- Souriau's Lie group statistical mechanics
- Souriau's thermodynamics of continua

Bridging the gap between both theories (1/5)

- **Step 1 : parameterizing the orbit.** Galileo's group is the set of affine transformations $t = t' + \tau_0,\ x = R\,x' + u\,t' + k$ where u is the Galilean boost

  The infinitesimal action $Z \cdot X$ is $\delta t = \delta\tau_0,\ \delta x = \delta\varpi \times x + \delta u\,t + \delta k$

  The dualing pairing is

$$\mu\,Z = l \cdot d\varpi - q \cdot du + p \cdot dk - e\,d\tau_0$$

  where l is the angular momentum, q the passage, p the linear momentum and e the energy

  In the dual space $\mathfrak{g}^*$ of dimension 10, the generic orbits are submanifolds parameterized by $(q, p, n) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$ where $n = l\ /\ \|\ l\ \|$

Bridging the gap between both theories (2/5)

- **Step 2 : modelling the deformation.** We consider N identical spinless particles in a box of volume V representing the elementary volume of the continuum thermodynamics

  For a coordinate change $t = t'$, $x = \varphi(t', s')$, the Jacobian matrix is

$$\frac{\partial X}{\partial X'} = P = \begin{pmatrix} 1 & 0 \\ v & F \end{pmatrix}$$

  If the box of initial volume V_0 is at rest $(v = 0)$ and the deformation gradient F is uniform in the box, $d\lambda$ is preserved

  Replacing the orbit by the subset $V_0 \times \mathbb{R}^3 \times \mathbb{S}^2$ and integrating gives

$$z = \frac{1}{2} \ln(\det(C)) - \frac{3}{2} \ln \beta + C^{te}$$

Bridging the gap between both theories (3/5 and 4/5)

- **Step 3 : boost method.** A new coordinate system $\bar{X}$ in which the box has the velocity v can be deduced from $X = P\,\bar{X} + C$ by applying a boost $u = -v$. Leaving out the bars, we have

$$z = \frac{1}{2}\ln(\det(C)) - \frac{3}{2}\ln\beta + \frac{m}{2\,\beta}\ \parallel w \parallel^2 + C^{te}\ .$$

- **Step 4 : identification. Theorem**
  The transformation law of the temperature vector $W = (\beta, w)$ and Planck's potential ζ is the same as the one of the components z, Z of the affine map $\boldsymbol{\Theta}$ through the identification

$$Z = (-W, 0), \qquad z = m\,\zeta$$

Bridging the gap between both theories (5/5)

- **Step 5 : from Planck's potential**

$$\zeta = \frac{z}{m} = \frac{1}{2\,m}\ \ln(\det(C)) - \frac{3}{2\,m}\ \ln\beta + \frac{1}{2\,\beta}\ \parallel w \parallel^2 + C^{te}$$

we deduce the linear 4-**momentum** $\Pi = (\mathcal{H}, -p^T)$ **and Cauchy's stresses**

$$\mathcal{H} = \rho\left(\frac{3}{2}\,\frac{k_B\,T}{m} + \frac{1}{2} \parallel v \parallel^2\right), \qquad p = \rho v, \qquad \sigma = -q\,1_{\mathbb{R}^3}$$

where we recover the **ideal gas law** $q = \frac{\rho}{m}\,k_B\,T = \frac{N}{V}\,k_B\,T$

Thank you !