Covariant Momentum Map Thermodynamics

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General Covariant Statistical Mechanics?

Statistical mechanics is inherently Hamiltonian: time and energy play a fundamental role in the definition of equilibrium. Can we still deal with statistical mechanics in a General Relativistic setting, with no absolute spacetime background? (Einstein's principle of Covariance)

- **information theory** allows to tackle the problem in terms of a reinterpretation of statistical mechanics based on statistical inference (partial knoweldge)

  e.g. Maximum entropy principle: [Jaynes, 1957]

  - macrostates of a system with many d.o.f. is defined by a finite number of observable averages $\{O_a\}$ on the phase space of the system M
  - the least biased statistical distribution compatible with our limited knowledge of the macroscopic system (in terms of $\{O_a\}$) is that which maximises the information entropy

- **geometry** offers an abstract yet intuitive tool to generalise and investigate key fundamental structures **e.g. Momentum Map**:

– relevant observable associated to the notion of equilibrium is indicated by the symmetry of the system, and specified by the momentum map associated to the Hamiltonian action of the dynamical group of the system on its space of solutions

e.g.

- Let N be the configuration manifold of a Lagrangian system $L : TN \to \mathbb{R}$. Let $H : T^*N \to \mathbb{R}$ be the corresponding Hamiltonian and (M, ω) be the symplectic manifold of motions, namely the reduced phase space of the system.
- the Hamiltonian $H : T^*N \to \mathbb{R}$ remains constant along each motion of the system $=>$ we can define on (M, ω) a smooth function $E : M \to \mathbb{R}$, called the *energy function*

$$E(\varphi) = H(\varphi(t)) \quad \text{for all } t \in R, \quad \varphi \in M.$$

in this standard case we have:

- the Hamiltonian vector field X_E on M is the infinitesimal generator of the 1-dimensional group of **time translations**. A time translation $\Delta t : \mathbb{R} \to \mathbb{R}$ is a map $\Delta t : \mathbb{R} \to \mathbb{R}$, $\Delta t(t) = t + \Delta t$.
- The group of time translations can be identified with $\mathbb{R}$. It acts on the manifold of motions M by the action Φ^E , such that for each time translation Δt and each motion φ, $\Phi^E_{\Delta t}(\varphi)$ is the motion

$$t \to \Phi^E_{\Delta t}(\varphi(t)) = \varphi(t + \Delta t).$$

- the energy function E on (M, ω) is the momentum map of the Hamiltonian action Φ^E on that manifold of the one-dimensional Lie group of time translations.

Generalization for a Hamiltonian Lie group action

more generally:

- Let Φ be a Hamiltonian action of a finite-dimensional Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω). $\exists$ smooth map $J : M \to \mathfrak{g}^*$ such that for each $Y \in \mathfrak{g}$ the Hamiltonian vector field ξ_Y on M admits as Hamiltonian the function $J_Y \equiv \langle J(x), Y \rangle : M \to \mathbb{R}$, defined by

$$\xi_Y \lrcorner \omega = \mathrm{d}J_Y.$$

  e.g. $SO(3)$ action on the sphere: angular momentum

- The (co)momentum map $\langle J(x), b \rangle$, with $b \in \mathfrak{g}$, provides a natural vector-valued generalisation of the Hamiltonian function

- Souriau's covariant Lie-group reformulation of statistical mechanics combines the two aspects: [Souriau, 69; Marle, de Saxcé, Barbaresco, Gay-Balmaz]

=> exploits a natural generalisation of the definition of a thermodynamic equilibrium state in which a Lie group G acts, by a Hamiltonian action Φ, on that symplectic manifold.

=> covariance achieved by shifting the setting to the fully reduced phase space

Lie group Thermodynamics (LGT)

A *statistical state* on (M, ω) is a probability law μ on M defined by the product of the Liouville density of M with a classical distribution function

$$\mu(\mathcal{A}) = \int_{\mathcal{A}} \rho(x)\, \omega^n(x)$$

for each Borel subset $\mathcal{A}$ of M, with $\rho : M \to \mathbb{R}\,([0, +\infty[)$ being a continuous density function, such that $\int_M \rho(x)\, \omega^n(x) = 1$.

Thermodynamic equilibria μ_{eq} maximise entropy with constraint $\mathbb{E}_\rho[J] = \mathit{const}$,

$$S(\rho) = -\int_M \rho(x)\, \log\left(\rho(x)\right) \omega^n(x)\,,$$

and such that $S(\rho_{eq})$ is stationary with respect to all infinitesimal smooth variations of the probability density. The results is the Gibbs distribution

$$\rho(\xi) = \frac{1}{Z(\xi)} e^{-\langle J(x), \xi \rangle},$$

covariant under the action of any one-parameter subgroup of G on M.

Motivating Questions

Generalise this symplectic statistical framework in two directions:

- formally, for field theories
- conceptually, accounting for diffeomorphism invariance, or general covariance symmetry.
- radically moving the whole construction off-shell, on the *generalised* phase space.

Generally covariant theories have no notion of distinguished physical time with respect to which everything evolves, while being completely characterised by a vanishing (super)Hamiltonian (momentum map) constraint.

E.g., in General Relativity, spacetime as a field does not evolve, but it can be viewed itself as the evolution, or history, of (three dimensional) space, *constrained* as to satisfy the Einstein equation.

Desired insights:

- a (timeless) notion of general-covariant Gibbs equilibrium
- investigate the effect of general covariance on Souriau's Lie group thermodynamics

Field Theory Parenthesis

A field (e.g. N-component real scalar field) is a map:

$$\varphi : B \to \mathbb{R}^N$$

A field theory is a variational problem, characterised by:

- a Lagrangian density

  $$L(x, \varphi(x), \mathrm{d}\varphi(x))dvol$$

  dvol the volume form on B

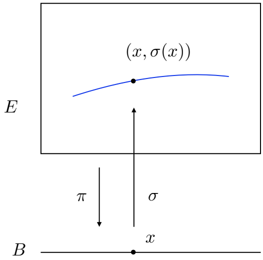
- and an action

  $$A[\varphi] = \int_B L(x, \varphi(x), \mathrm{d}_x\varphi(x))dvol$$

Field Theory Parenthesis

In a bundle description, a section/field gives an abstract characterisation of a graph of $g\colon B \to Y$, identified with a function taking its values in the Cartesian product $E = B \times Y$:

$$\sigma\colon B \to E, \quad \sigma(x) = (x, g(x)) \in E.$$



e.g.: a vector **field** on a smooth manifold $\mathcal{X}$ is a choice of tangent vector at each point of $\mathcal{X}$: this is a section of the tangent bundle of $\mathcal{X}$.
(*statistical bundle* on next talk w/ Malagò)

Schematic Outline

1. introduce a multisymplectic description of 1st-order field theories
2. define a covariant extension of the field-theoretic setting, via parametrization
3. reproduce the LGT for an off shell multi-phase space of fields (and conjugate momenta), w.r.t. the hamiltonian action associated to a one-parameter family of spacetime diffeomorphisms

Notation from : [GIMM 98, Hélein et al. 02, Cariñena 91, therein refs.]

Let B be an oriented $(n+1)$-dimensional manifold $\mathcal{X}$ (spacetime), and $\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$ a finite-dimensional fiber bundle over $\mathcal{X}$ with fibers $\mathcal{Y}_x$ over $x \in \mathcal{X}$ of dimension N.

The fiber bundle $\mathcal{Y}$ is the field theoretic analogue of the configuration space in classical mechanics, or **configuration bundle**. Physical fields correspond to sections ϕ of the bundle.

The Lagrangian density for a first order classical field theory is given by

$$\mathscr{L} : J^1(\mathcal{Y}) \longrightarrow \Lambda^{n+1}(\mathcal{X}) \, ,$$

where $J^1(\mathcal{Y})$ is the first jet bundle[1] of $\mathcal{Y}$, the field-theoretic analogue of the **tangent bundle** of classical mechanics, and $\Lambda^{n+1}(\mathcal{X})$ is the space of the $(n+1)$-forms on $\mathcal{X}$.

Coordinates

A set of local coordinates (x^μ, y^A) on $\mathcal{Y}$ consists in $n+1$ local coordinates x^μ, $\mu = 0, \ldots, n$ on $\mathcal{X}$, and N fiber coordinates y^A, $A = 1, \ldots, N$, which give the components of the field at a given point $x \in \mathcal{X}$.

Local coordinates (x^μ, y^A) on $\mathcal{Y}$ induce coordinates v^A_μ on the fibers of $J^1(\mathcal{Y})$, so that the first jet prolongation $j^1\phi$ of a section ϕ of the bundle $\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$ gives

$$j^1\phi : x^\mu \longmapsto (x^\mu, y^A, v^A_\mu) = (x^\mu, y^A(x), y^A_{,\mu}(x)) \; ,$$

where $y^A_{,\mu} = \partial_\mu y^A$ and $\partial_\mu = \partial/\partial x^\mu$.

The Lagrangian then reads

$$\mathscr{L}(j^1\phi) = L\left(x^\mu, y^A(x), y^A_{,\mu}(x)\right) \mathrm{d}^{n+1}x \; ,$$

where $\mathrm{d}^{n+1}x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$ is the volume form on $\mathcal{X}$.

Via the Legendre transform, we introduce multimomenta p_A^μ and *covariant Hamiltonian* p,

$$p_A^\mu = \frac{\partial L}{\partial v_\mu^A} \qquad , \qquad p = L - \frac{\partial L}{\partial v_\mu^A} v_\mu^A \ ,$$

leads to the field-theoretic analogue of the phase space of classical mechanics, which provided by the so-called **multiphase space** $\mathcal{Z}$.

The space $\mathcal{Z}$ is canonically isomorphic to the dual jet bundle $J^1(\mathcal{Y})^*$, the latter playing the role of the field-theoretic analogue of the cotangent bundle.
[Gotay, Marsden et al.]

The **canonical Poincaré-Cartan** $(n+1)$**-form** Θ and $(n+2)$**-form** Ω on $\mathcal{Z}$ is given by,

$$\begin{aligned}
\Theta &= p\,\mathrm{d}^{n+1}x + p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu \ , \\
\Omega &= -\mathrm{d}\Theta = \mathrm{d}y^A \wedge \mathrm{d}p_A^\mu \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^{n+1}x \ .
\end{aligned}$$

The pair $(\mathcal{Z}, \Omega)$ is an example of multisymplectic manifold.

Multisymplectic Formulation: Field Theory / Classical Mechanics

Classical Mechanics ($n = 0, \mathcal{X} \equiv \mathbb{R}$)	**Field Theory** ($n > 0, \dim \mathcal{X} = n+1$)
extended configuration space $\mathcal{Y} = \mathbb{R} \times \mathcal{Q}$	configuration bundle over spacetime $\mathcal{Y} \xrightarrow{\pi_{\mathcal{XY}}} \mathcal{X}$
local coordinates on $\mathcal{Y}$ (t, q^A)	local coordinates on $\mathcal{Y}$ (x^μ, y^A)
extended phase space $\mathcal{P} = T^*\mathcal{Y} = T^*\mathbb{R} \times T^*\mathcal{Q}$	multiphase space $J^1(\mathcal{Y})^* \cong \mathcal{Z} \subset \Lambda^{n+1}(\mathcal{Y})$
local coordinates on $\mathcal{P}$ (t, q^A, E, p_A)	local coordinates on $\mathcal{Z}$ (x^μ, y^A, p, p^μ_A)
Poincaré-Cartan 1-form on $\mathcal{P}$ $\Theta = p_A \mathrm{d}q^A + E\mathrm{d}t$	Poincaré-Cartan (n+1)-form on $\mathcal{Z}$ $\Theta = p\mathrm{d}^{n+1}x + p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu$
symplectic 2-form on $\mathcal{P}$ $\Omega = \mathrm{d}q^A \wedge \mathrm{d}p_A - \mathrm{d}E \wedge \mathrm{d}t$	multisymplectic (n+2)-form on $\mathcal{Z}$ $\Omega = \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^{n+1}x$

General Covariant Extension (no fixed background)

We shall make the theory generally covariant, i.e., with the spacetime diffeomorphism group as symmetry group, by introducing diffeomorphisms as new dynamical fields, while leaving the solution space unchanged.
[Castrillón López, Gotay and Marsden]

$$
\begin{tikzcd}
\mathcal{Y} \times_{\mathcal{X}} (\tilde{\mathcal{X}} \times \mathcal{X}) \arrow[d, "\pi_{\mathcal{X}\tilde{\mathcal{Y}}}"] \\
\mathcal{X} \arrow[u, dashed, bend left=60, "\phi"] \arrow[u, dashed, bend right=60, "\eta"']
\end{tikzcd}
$$

- introduce **covariance fields**: (oriented) diffeomorphisms of $\mathcal{X}$, reinterpreted as sections $\eta : \mathcal{X} \to \tilde{\mathcal{X}}$ of the bundle $\tilde{\mathcal{X}} \times \mathcal{X} \xrightarrow{\tilde{\pi}} \mathcal{X}$, where $\tilde{\mathcal{X}}$ is a copy the base manifold equipped with a given metric g.
- the configuration bundle $\mathcal{Y}$ is extended to the fibered product $\tilde{\mathcal{Y}} = \mathcal{Y} \times_{\mathcal{X}} (\tilde{\mathcal{X}} \times \mathcal{X})$, with sections (ϕ, η).

Diffeomorphism-Covariant Formulation

- the Lagrangian density $\mathscr{L}$ of the theory is replace by a new density

$$\tilde{\mathscr{L}} : J^1(\tilde{\mathcal{Y}}) \longrightarrow \Lambda^{n+1}(\mathcal{X}) ,$$

  such that $\tilde{\mathscr{L}}(j^1\phi, j^1\eta) := \mathscr{L}(j^1\phi, \eta^* g)$.
- for $(x^\mu, y^A, v^A_\mu, u^a, u^a_\mu)$ coordinates on $J^1(\tilde{\mathcal{Y}})$ we have

$$\tilde{\mathscr{L}}(x^\mu, y^A, v^A_\mu, u^a, u^a_\mu) = \mathscr{L}(x^\mu, y^A, v^A_\mu ; G_{\mu\nu}) ,$$

  where

$$G_{\mu\nu} \equiv (\eta^* g)_{\mu\nu} = \eta^a_{,\mu}\, \eta^b_{,\nu}\, g_{ab}; .$$

- the fixed background metric g is no longer thought of as living on $\mathcal{X}$, but rather as a geometric object on the copy $\tilde{\mathcal{X}}$ in the fiber of the extended configuration bundle $\tilde{\mathcal{Y}}$. The pulled back metric variable $G = \eta^* g$ on $\mathcal{X}$ inherits a dynamical character from the covariance field η.

The true dynamical fields of the parametrized theory are then given by ϕ and η.

Diffeomorphisms on the Parametrised Setting

Consider a diffeomorphism of the base, $\alpha_{\mathcal{X}} \in \mathrm{Diff}(\mathcal{X})$, and $\alpha_{\mathcal{Y}} \in \mathrm{Aut}(\mathcal{Y})$ its lift to $\mathcal{Y}$. Its action is extended to an action on $\tilde{\mathcal{Y}}$ via bundle automorphisms by requiring that $\mathrm{Diff}(\mathcal{X})$ acts trivially on $\tilde{\mathcal{X}}$, i.e.

$$\alpha_{\tilde{\mathcal{X}}} : \tilde{\mathcal{X}} \times \mathcal{X} \longrightarrow \tilde{\mathcal{X}} \times \mathcal{X} \qquad \text{by} \qquad (u, x) \longmapsto (u, \alpha_{\mathcal{X}}(x)) \, .$$

The induced action on the space $\tilde{\mathscr{Y}} \equiv \Gamma(\mathcal{X}, \tilde{\mathcal{Y}})$ of sections of $\tilde{\mathcal{Y}}$ is

$$\alpha_{\tilde{\mathscr{Y}}}(\phi, \eta) = (\alpha_{\mathscr{Y}}(\phi), \alpha_{\tilde{\mathscr{X}}}(\eta)) \, ,$$

where $\alpha_{\mathscr{Y}}(\phi) = \alpha_{\mathcal{Y}} \circ \phi \circ \alpha_{\mathcal{X}}^{-1}$, for $\phi \in \mathscr{Y} \equiv \Gamma(\mathcal{X}, \mathcal{Y})$, generalizes the usual push-forward action on tensor fields, and

$$\alpha_{\tilde{\mathscr{X}}}(\eta) = \eta \circ \alpha_{\mathcal{X}}^{-1} \, ,$$

is the (left) action by composition on sections of the trivial bundle $\tilde{\mathcal{X}} \times \mathcal{X}$.

The modified field theory on $J^1(\tilde{\mathcal{Y}})$ is **Diff($\mathcal{X}$)-covariant**. The Lagrangian density is **Diff($\mathcal{X}$)-equivariant**, [Castrillón López, Gotay and Marsden]

$$\tilde{\mathscr{L}}\left(j^1(\alpha_{\mathscr{Y}}(\phi)), j^1(\alpha_{\tilde{\mathscr{X}}}(\eta))\right) = (\alpha_{\mathcal{X}}^{-1})^* \left[\tilde{\mathscr{L}}(j^1\phi, j^1\eta)\right] .$$

Diffeomorphism-Covariant Multi Phase Space

The covariant multiphase space $\tilde{\mathscr{Z}} \cong J^1(\tilde{\mathcal{Y}})^*$ equipped with a canonical Poincaré-Cartan $(n+1)$-form

$$\tilde{\Theta} = \tilde{p}\,\mathrm{d}^{n+1}x + p^{\mu}_{A}\mathrm{d}y^{A} \wedge \mathrm{d}^{n}x_{\mu} + \varrho^{\mu}_{a}\mathrm{d}u^{a} \wedge \mathrm{d}^{n}x_{\mu}$$

the covariant Hamiltonian $\tilde{p}$ and the multimomenta $p^{\mu}_{A}, \varrho^{\mu}_{a}$ (respectively conjugate to the multivelocities v^{A}_{μ} and u^{a}_{μ}) are defined w.r.t. $\tilde{\mathscr{L}}$ via Legendre transform

The multisymplectic $(n+2)$-form $\tilde{\Omega} = -\mathrm{d}\tilde{\Theta}$ on $\tilde{\mathscr{Z}}$ then reads

$$\tilde{\Omega} = \mathrm{d}y^{A} \wedge \mathrm{d}p^{\mu}_{A} \wedge \mathrm{d}^{n}x_{\mu} + \mathrm{d}u^{a} \wedge \mathrm{d}\varrho^{\mu}_{a} \wedge \mathrm{d}^{n}x_{\mu} - \mathrm{d}\tilde{p} \wedge \mathrm{d}^{n+1}x$$

Such extended multisymplectic formalism generalise the symplectic framework of LGT for our generally covariant field theory setting, in an off-shell framework. Now as a Lie group $\mathcal{G}$, we shall take a subgroup of $\mathrm{Aut}(\tilde{\mathcal{Y}})$ covering diffeomorphisms on $\mathcal{X}$.

Diffeomorphism-Covariant Multi Phase Space

Given an element $\xi \in \mathfrak{g}$, denote by $\xi_{\mathcal{X}}, \xi_{\mathcal{Y}}, \xi_{\tilde{\mathcal{Y}}}$, and $\xi_{\tilde{\mathcal{Z}}}$ the infinitesimal generators of the corresponding transformations on $\mathcal{X}, \mathcal{Y}, \tilde{\mathcal{Y}}$, and $\tilde{\mathcal{Z}}$

The group $\mathcal{G}$ is said to act on $\tilde{\mathcal{Z}}$ by **covariant canonical transformation** if the $\mathcal{G}$-action corresponds to an infinitesimal multi-symplectomorphism, i.e.

$$\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}} \tilde{\Omega} = 0 \ ,$$

where $\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}}$ denotes the Lie derivative along $\xi_{\tilde{\mathcal{Z}}}$

A covariant momentum map associated to the action of $\mathcal{G}$ on $\tilde{\mathcal{Z}}$ by covariant canonical transformations is a map

$$\tilde{\mathcal{J}} : \tilde{\mathcal{Z}} \longrightarrow \mathfrak{g}^* \otimes \Lambda^n(\tilde{\mathcal{Z}}) \ ,$$

given by

$$\mathrm{d}\tilde{\mathcal{J}}(\xi) = \mathrm{i}_{\xi_{\tilde{\mathcal{Z}}}} \tilde{\Omega},$$

where $\tilde{\mathcal{J}}(\xi)$ is the n-form on $\tilde{\mathcal{Z}}$ whose value at $\tilde{z} \in \tilde{\mathcal{Z}}$ is $\langle \tilde{\mathcal{J}}(\tilde{z}), \xi \rangle$ with $\langle \cdot, \cdot \rangle$ being the pairing between the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

Covariant Momentum Map as a Statistical Observable

Two important remarks:

- the covariant (co)momentum map provides us with the right general-covariant extension of "Hamiltonian". Despite the fully covariant setting, the hamiltonian character is associated to a reference frame choice $=>$ reduction codimension-one forms
- the multimomentum map $J : \tilde{\mathcal{Z}} \to \mathfrak{g}^* \otimes \Lambda^n(\tilde{\mathcal{Z}})$ has a differential form component and the definition of statistical averages turns out to be subtle.

The covariant field-theoretic formalism extension requires a further step: n-forms on $\mathcal{Z}$ should be integrated over hypersurfaces of $\mathcal{Z}$ in order to produce proper **functionals**.

- we need to introduce a spacetime foliation, while keeping the formalism *covariant*.
- we move to a (n+1) (covariantised) canonical formalism (alternatively use Hélein & Kouneiher covariant approach)

General-Covariance via $\mathcal{G}$-Slicing

- A foliation $\mathfrak{s}_{\mathcal{X}}$ of $\mathcal{X}$ corresponds to a 1-parameter family $\lambda \mapsto \tau_\lambda$ of space-like embeddings $\tau_\lambda \in \mathsf{Emb}_G(\Sigma, \mathcal{X})$ of Σ in $\mathcal{X}$, i.e.

$$\mathfrak{s}_{\mathcal{X}} : \Sigma \times \mathbb{R} \to \mathcal{X} \quad \text{by} \quad (\vec{x}, \lambda) \mapsto \mathfrak{s}_{\mathcal{X}}(\vec{x}, \lambda)\,,$$

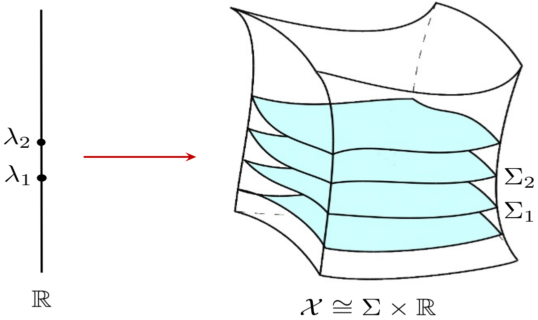
  such that

$$\tau \equiv \tau_\lambda : \Sigma \to \mathcal{X} \quad \text{by} \quad \tau(\vec{x}) \equiv \tau_\lambda(\vec{x}) := \mathfrak{s}_{\mathcal{X}}(\vec{x}, \lambda)\ ,$$

- The foliation of $\mathcal{X}$ induces a *compatible slicing* of bundles over it whose generating vector fields project to $\zeta_{\mathcal{X}}$.

- We consider a $\mathcal{G}$-**slicing** in which case the one-parameter group of automorphisms of the extended configuration bundle is induced by a one parameter subgroup of the gauge group $\mathcal{G}$, i.e., $\zeta_{\tilde{\mathfrak{Y}}} = \xi_{\tilde{\mathfrak{Y}}}$ for some $\xi \in \mathfrak{g}$.

General-Covariance via $\mathcal{G}$-Slicing



Spatial fields are identified with smooth sections of the pull-back bundle

$$\mathcal{Y}_\tau \to \Sigma_\tau$$

over a Cauchy surface given by $\varphi := \phi_\tau = \tau^* \phi$

Embedding Fields

- The multisymplectic structure on $\tilde{\mathcal{Z}}$ induces a presymplectic structure on the space $\tilde{\mathscr{Z}}_\tau$ of sections of the bundle $\tilde{\mathcal{Z}}_\tau \to \Sigma_\tau$.
- The parametrization procedure makes the embeddings $\tau \in \mathrm{Emb}_G(\Sigma, \mathcal{X})$ dynamical via the covariance fields η. We have

$$\tau = \eta^{-1} \circ \tilde{\tau} \quad (\textit{dressing})$$

  for a given space-like embedding $\tilde{\tau} \in \mathrm{Emb}_g(\Sigma, \tilde{\mathcal{X}})$ of Σ into $\tilde{\mathcal{X}}$ associated to the slicing of $\tilde{\mathcal{X}}$ w.r.t. the fixed metric g.

The new canonical parametrized configuration space consists of the pairs

$$\sigma = (\varphi, \tau)$$

of spatial fields defined over a Cauchy slice and the space-like embeddings identifying a $\mathcal{G}$-slicing of spacetime w.r.t. one-parameter subgroups of diffeomorphisms.

Parametrised Energy Momentum Map

Let now $\sigma \in \tilde{\mathscr{Z}} \equiv \Gamma(\mathcal{X}, \tilde{\mathcal{Z}})$ be a section of the bundle $\tilde{\mathcal{Z}}$ over $\mathcal{X}$, and let $\alpha_{\tilde{\mathcal{Z}}} : \tilde{\mathcal{Z}} \to \tilde{\mathcal{Z}}$ be a covariant canonical transformation covering a diffeomorphism $\alpha_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ whose induced action on sections is given by $\alpha_{\tilde{\mathscr{Z}}}(\sigma) = \alpha_{\tilde{\mathcal{Z}}} \circ \sigma \circ \alpha_{\mathcal{X}}^{-1}$.
The corresponding transformation on $\tilde{\mathscr{Z}}_\tau \equiv \Gamma(\Sigma_\tau, \tilde{\mathscr{Z}})$ given by

$$\begin{aligned} \alpha_{\tilde{\mathscr{Z}}_\tau} : \tilde{\mathscr{Z}}_{\eta^{-1}\circ\tilde{\tau}} &\longrightarrow \tilde{\mathscr{Z}}_{\alpha_{\tilde{\mathscr{X}}}(\eta)^{-1}\circ\tilde{\tau}} \\ \sigma &\longmapsto \alpha_{\tilde{\mathscr{Z}}_\tau}(\sigma) = \alpha_{\tilde{\mathcal{Z}}} \circ \sigma \circ \alpha_\tau^{-1} \ , \end{aligned}$$

Theorem

*The covariant multimomentum map associated to the $\mathcal{G}$-action on $\tilde{\mathcal{Z}}$ induces a (**parametrized**) **energy-momentum map**, $\tilde{\mathcal{E}}_\tau : \tilde{\mathscr{Z}}_\tau \longrightarrow \mathfrak{g}^*$, defined by*

$$\tilde{\mathcal{E}}_\tau(\sigma, \eta) = \tilde{\mathcal{E}}_{\eta^{-1}\circ\tilde{\tau}}(\sigma) := \int_{\Sigma_\tau} \sigma^* \langle \tilde{\mathcal{J}}, \xi \rangle \ ,$$

is Ad-equivariant w.r.t. the $\alpha_{\tilde{\mathscr{Z}}_\tau}$ action, we have*

$$\langle \tilde{\mathcal{E}}_\tau(\sigma, \eta), Ad_\alpha^{-1}\xi \rangle = \langle \alpha^*_{\tilde{\mathscr{Z}}_\tau}\left[\tilde{\mathcal{E}}_\tau(\sigma, \eta)\right], \xi \rangle \ .$$

Parametrised Energy Momentum Map

Proof.

$$
\begin{aligned}
\langle \alpha^*_{\tilde{\mathscr{Z}}_\tau}\left[\tilde{\mathcal{E}}_\tau(\sigma,\eta)\right], \xi\rangle &= \langle \tilde{\mathcal{E}}_{\alpha_{\tilde{\mathscr{X}}}(\eta)^{-1}\circ\tilde{\tau}}(\alpha_{\tilde{\mathscr{Z}}_\tau}(\sigma))\rangle \\
&= \int_{(\eta\circ\alpha_{\mathcal{X}}^{-1})^{-1}\circ\tilde{\tau}(\Sigma)} (\alpha_{\tilde{\mathcal{Z}}}\circ\sigma\circ\alpha_\tau^{-1})^* \langle \tilde{\mathcal{J}}, \xi\rangle \\
&= \int_{\alpha_{\mathcal{X}}\circ(\eta^{-1}\circ\tilde{\tau})(\Sigma)} (\alpha_\tau^{-1})^*\, \sigma^*\, \alpha^*_{\tilde{\mathcal{Z}}} \langle \tilde{\mathcal{J}}, \xi\rangle \\
&= \int_{\eta^{-1}\circ\tilde{\tau}(\Sigma)} \sigma^*\, \alpha^*_{\tilde{\mathcal{Z}}} \langle \tilde{\mathcal{J}}, \xi\rangle \qquad \text{(change of variables)} \\
&= \int_{\eta^{-1}\circ\tilde{\tau}(\Sigma)} \sigma^*\, \langle \tilde{\mathcal{J}}, \mathrm{Ad}_\alpha^{-1}\xi\rangle \qquad (\mathrm{Ad}^*\text{-equivariance of } \tilde{\mathcal{J}}) \\
&= \langle \tilde{\mathcal{E}}_{\eta^{-1}\circ\tilde{\tau}}(\sigma), \mathrm{Ad}_\alpha^{-1}\xi\rangle \\
&= \langle \tilde{\mathcal{E}}_\tau(\sigma,\eta), \mathrm{Ad}_\alpha^{-1}\xi\rangle \ .
\end{aligned}
$$

□

Disentangling the Information in $\tilde{\mathcal{E}}_\tau$

We shall now restrict our analysis on the symplectic space given by the canonical parametrized phase space $T^*\tilde{\mathscr{Y}}_\tau$, isomorphic to the quotient

$$\tilde{\mathscr{Z}}_\tau \,/\, \ker \tilde{\Omega}_\tau .$$

This allows us to express $\tilde{\mathcal{E}}_\tau$ in connection with the initial value constraints, hence to partially disentangle the information about gauge and dynamics for the given classical field theory.

- we denote (φ, Π, τ, P) a point in the canonical parametrised phase space

  $$T^*\tilde{\mathscr{Y}}_\tau = T^*\mathscr{Y}_\tau \times T^*\mathrm{Emb}_G(\Sigma, \mathcal{X})$$

  with canonical symplectic structure $\tilde{\omega}_\tau$.

- we focus on $\tilde{\mathscr{P}}_\tau$, the primary constraint submanifold in $T^*\tilde{\mathscr{Y}}_\tau$ defined as $\tilde{\mathscr{P}}_\tau = R_\tau(\tilde{\mathscr{N}}_\tau) \subset T^*\tilde{\mathscr{Y}}_\tau$ with $\tilde{\mathscr{N}}_\tau = \mathbb{F}\mathscr{L}((j^1\tilde{\mathscr{Y}})_\tau) \subset \tilde{\mathscr{Z}}_\tau$, $\mathbb{F}\mathscr{L}$ being the Legendre transform.

Disentangling the Information in $\tilde{\mathcal{E}}_\tau$

- The projection of the parametrized energy-momentum map $\tilde{\mathcal{E}}_\tau$ on the primary constraint submanifold is $\tilde{\mathcal{J}}_{H:\tilde{\mathscr{P}}_\tau \longmapsto \mathfrak{g}^*}$, given by

$$
\begin{aligned}
\langle \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P), \zeta \rangle &= -\int_{\Sigma_\tau} \mathrm{d}^n x_0 (\zeta^\mu \mathcal{H}^{(\varphi)}_\mu + \zeta^\mu P_\mu) \\
&= -\left(H^{(\varphi)}(\zeta)(\varphi, \Pi, \tau) + P(\zeta)(\tau, P) \right) ,
\end{aligned}
$$

  where $P_\mu = \eta^a_{,\mu} P_a$ is the pull-back of P_a to Σ along η.

- $\tilde{\mathcal{J}}_H$ encodes the first-class secondary constraints in its transverse and tangential components to the spatial slice (super-momenta and Hamiltonian constraints).

We are now ready to define a Lie-group generalization of the Gibbs equilibrium state on the parametrized phase space is $\Upsilon \equiv T^*\mathscr{Y}_\tau \times T^*\mathrm{Emb}_G(\Sigma, \mathcal{X})$ in terms of the Hamiltonian action of the spacetime diffeomorphism group $\mathrm{Diff}(\mathcal{X})$.

- Given a functional on Υ, say $f \in \mathcal{F}(\Upsilon)$, the expected value of f w.r.t. ρ is now well defined as

$$\mathbb{E}_\rho[f] = \int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P]\rho(\varphi, \Pi, \tau, P) f(\varphi, \Pi, \tau, P) : \mathcal{F}(\Upsilon) \longrightarrow \mathbb{R} \ .$$

- A statistical state $\rho : \Upsilon \to \mathbb{R}([0, +\infty[)$ on Υ, is a smooth probability density on Υ such that, for any Borel subset $\mathscr{A}$ of Υ, the integral

$$\mu(\mathscr{A}) = \int_{\mathscr{A}} \mathcal{D}[\varphi, \Pi, \tau, P] \, \rho(\varphi, \Pi, \tau, P)$$

defines a probability measure on Υ with the normalization condition

$$Z(\rho) = \int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P] \, \rho(\varphi, \Pi, \tau, P) = 1 \ ,$$

where $\mathcal{D}[\varphi, \Pi, \tau, P]$ formally denotes the integration measure on Υ (assumed to be $\mathrm{Diff}(\mathcal{X})$-invariant).

Maximum Entropy Principle

To such a statistical state, we can associate a entropy functional

$$S(\rho) = -\int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P]\, \rho(\varphi, \Pi, \tau, P) \log \rho(\varphi, \Pi, \tau, P)\ ,$$

We ask for the stationarity of the entropy functional under infinitesimal smooth variation $\rho_s(\varphi, \Pi, \tau, P)$ with $s \in]-\varepsilon, \varepsilon[$, $\varepsilon > 0$ of the statistical state ρ, for fixed expected value of the co-momentum map, namely

$$\mathbb{E}_\rho(\tilde{\mathcal{J}}_H(\xi)) = \int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P]\, \rho(\varphi, \Pi, \tau, P)\, \langle \xi(\tau), \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P) \rangle = const\ .$$

This is implemented by introducing two real Lagrange multipliers $a, b \in \mathbb{R}$ respectively associated to the standard normalization condition and to the constraint $\mathbb{E}_\rho(\tilde{\mathcal{J}}_H(\xi)) = const$, via

$$\mathcal{S}(\rho_s) = S(\rho_s) + b\, \mathbb{E}_{\rho_s}(\tilde{\mathcal{J}}_H(\xi)) + a\, Z(\rho_s)\ ,$$

such that

$$\left.\frac{\delta \mathcal{S}(\rho_s)}{\delta s}\right|_{s=0} = 0 \qquad \forall\ \rho_s\ .$$

Maximum Entropy Principle

For any ρ_s, we have

$$0 = \left.\frac{\delta \mathcal{S}(\rho_s)}{\delta s}\right|_{s=0}$$
$$= -\int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P] \Big(1 + \log\left(\rho(\varphi, \Pi, \tau, P)\right) - b\, \tilde{\mathcal{J}}_H(\xi)(\varphi, \Pi, \tau, P) - a \Big) \left.\frac{\delta \rho_s}{\delta s}\right|_{s=0},$$

from which we get

$$\rho_{a,b}(\varphi, \Pi, \tau, P) = \exp\Big(-1 + a + b\, \langle \xi(\tau), \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P) \rangle \Big).$$

The normalization condition implies

$$Z(\xi, b) = \exp\left(1 - a\right) = \int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P]\, \exp\Big(b\, \tilde{\mathcal{J}}_H(\xi)(\varphi, \Pi, \tau, P) \Big),$$

where we limit b to the set of values such that the above integral *converge*.

Generally Covariant Gibbs State

The generally covariant Gibbs statistical state is given by

$$\rho^{(\mathbf{eq})}_{\xi_{(b)}}(\varphi,\Pi,\tau,P) = \frac{1}{Z(\xi,b)}\exp\left(b\,\langle\xi,\tilde{\mathcal{J}}_H(\varphi,\Pi,\tau,P)\rangle\right)$$

$$= \frac{1}{Z(\xi,b)}\exp\left(-\int_\Sigma \mathrm{d}^n x_0\,\xi^\mu_{(b)}(\tau(\vec{x}))\left(\mathcal{H}^{(\varphi)}_\mu(\vec{x}) + P_\mu(\vec{x})\right)\right)$$

- The dependence from the spacetime coordinates occurs only through the dynamical variables thus respecting the coordinate-independence of relativistic theories.
- As a functional of the embeddings, $\rho^{(\mathbf{eq})}$ is covariant in the sense that the momentum map is evaluated on any space-like hyper-surface without fixing the slicing a priori.
- The one-parameter group of automorphisms of the extended configuration space generated by $\xi_{(b)} \in \mathrm{diff}(\mathcal{X})$ identifies a generalized concept of "time evolution" w.r.t. which the Gibbs state is of equilibrium.

Remarks

The result is close in form to Souriau's generalised Gibbs state, but different in a key aspect due to the field-theoretic generalisation and the general covariance:

- In LGT, the variational principle requires the momentum map associated to the one-parameter group $\mathcal{G}$ action to be constant. Given that

$$\mathbb{E}_\rho(\tilde{\mathcal{J}}_H(\xi)) = \langle \xi, \mathbb{E}_\rho(\tilde{\mathcal{J}}_H) \rangle \ ,$$

  one can use the element of the algebra $\xi \in \mathrm{diff}(\mathcal{X})$ as the Lagrangian multiplier which eventually is identified with a global (inverse) temperature vector.

- Differently, in our field-theoretic approach, the necessity to integrate fields on slices implies that

$$\mathbb{E}_\rho(\tilde{\mathcal{J}}_H(\xi)) \neq \langle \xi, \mathbb{E}_\rho(\tilde{\mathcal{J}}_H) \rangle$$

Remarks

In relation to general covariance, we see that the dependence of the algebra realization on the embeddings makes Souriau's geometric temperature local.

- We can think of the covariant Gibbs state as the result of the exponentiation of a *local* thermodynamic equilibrium state w.r.t. a constant b,

$$\left(\exp\left(\langle \xi, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle\right)\right)^b$$

  multiplied by a b dependent factor that preserves the normalization: $\rho_b = Z^{-1}(b)\, \rho^b$ [Rovelli].

- The effect of this exponentiation is to scale the temperature globally.

- The product $\xi_{(b)} = b\,\xi \in \mathrm{diff}(\mathcal{X})$ defines a globally scaled vector field on $\mathcal{X}$ generating a one-parameter family of spacetime diffeomorphisms (scaled thermal time).

Thermodynamic Potentials

Given $Z(\xi, b)$, the free energy potential, $F(\xi, b) \equiv -\log Z(\xi, b)$, encodes complete thermodynamic information about the system. The *equilibrium* internal energy $Q : \Omega \to \mathbb{R}$ is given by the gradient of $F(b)$,

$$\begin{aligned} Q(\xi, b) &= -\partial_b(\log Z(\xi, b)) \\ &= -\int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P]\rho_{\xi_{(b)}}(\varphi, \Pi, \tau, P)\, \tilde{\mathcal{J}}_H(\xi)(\varphi, \Pi, \tau, P) \\ &= -\mathbb{E}_{\rho_{\xi_{(b)}}}[\tilde{\mathcal{J}}_H(\xi)] = \mathbb{E}_{\rho_{\xi_{(b)}}}[H^{(\varphi)}(\xi)] + \mathbb{E}_{\rho_{\xi_{(b)}}}[P(\xi)] \end{aligned}$$

corresponding to the average co-momentum map in the generalised Gibbs ensemble. Note that the internal energy is composed of two contributions, respectively associated to fields and embeddings.
The entropy function $S : \Omega \to \mathbb{R}$ has a strict maximum $S(b)$ at equilibrium,

$$\begin{aligned} S(\xi, b) &= -F(\xi, b) - b\, \mathbb{E}_{\rho_{\xi_{(b)}}}[\tilde{\mathcal{J}}_H(\xi)] \\ &= -F(\xi, b) + b\, \mathbb{E}_{\rho_{\xi_{(b)}}}[H^{(\varphi)}(\xi)] + b\, \mathbb{E}_{\rho_{\xi_{(b)}}}[P(\xi)] \end{aligned}$$

Note that, at equilibrium, the equivariance of the thermodynamic potentials is preserved.

We evolve the system from one equilibrium. This can be achieved either via a isothermal transformation, which leave the global temperature unchanged while moving the system on a different gauge orbit ($\xi_i \neq \xi_f$), or via a adiabathic transformation, corresponding to an overall shift along the same gauge orbit.

A convenient measure of the difference between initial and final equilibrium states is given by Kullback-Leibler divergence, which in our setting reads

$$D(\rho_i|\rho_f) = \int_\Upsilon \mathcal{D}[\sigma]\rho_i(\sigma)\log\left(\frac{\rho_i(\sigma)}{\rho_f(\sigma)}\right) \geq 0 \ ,$$

where σ synthetically denotes the elements in Υ as to ease the notation.

Isothermal Transformation

For the isothermal case, we have

$$\begin{aligned} D(\rho_{i,b}|\rho_{f,b}) &= \int_\Upsilon \mathcal{D}[\sigma]\rho_{i,b}(\sigma)\log\left(\frac{e^{b\,\tilde{\mathcal{J}}_H(\xi_i)}/Z_i(b)}{e^{b\,\tilde{\mathcal{J}}_H(\xi_f)}/Z_f(b)}\right) \\ &= -\log Z_i(b) + b\,\mathbb{E}_{\rho_{i,b}}[\tilde{\mathcal{J}}_H(\xi_i)] + \log Z_f(b) - b\,\mathbb{E}_{\rho_{i,b}}[\tilde{\mathcal{J}}_H(\xi_f)] \\ &= \log Z_f(b) - \log Z_i(b) - b\,\mathbb{E}_{\rho_{i,b}}[\tilde{\mathcal{J}}_H(\xi_f) - \tilde{\mathcal{J}}_H(\xi_i)] \geq 0 \end{aligned}$$

By interpreting $W \equiv -\left(\tilde{\mathcal{J}}_H(\xi_f) - \tilde{\mathcal{J}}_H(\xi_i)\right)$ as the work associated to the change of the element of the Lie algebra $\xi_i \to \xi_f$, we have

$$b\,\mathbb{E}_{\rho_{i,b}}(W) \geq \Delta F \equiv F_f(b) - F_i(b)\ .$$

The positivity of the divergence tells us that the external work performed on the system is no less than the free energy difference between the initial and final state.

Isothermal Transformation

- Notice that, at the infinitesimal level, the work is noting but the co-momentum map for the Lie-bracket, namely $W = -\tilde{\mathcal{J}}_H([\xi_f, \xi_i])$.

- Given the equivariance of the co-momentum map, this is the Poisson-bracket between the two co-momentum map, and we have

$$\mathbf{W} = -\{\tilde{\mathcal{J}}_{\mathbf{H}}(\xi_{\mathbf{i}}), \tilde{\mathcal{J}}_{\mathbf{H}}(\xi_{\mathbf{f}})\}$$

- When the parameter is varied slowly enough that the system remains in equilibrium along the flow, then the process is reversible and isothermal, and $\mathbb{E}_{i,b}(W) = \Delta F$.

- The case where the same two states at $[i, b]$ and $[f, b]$ are related via the action of a $\mathcal{G}$-slicing preserving one-d diffeomorphism subgroup is apparent: moving along an coadjoint orbit in $\mathfrak{g}^*$, by definition, we have $\Delta F = 0$.

Adiabatic Transformation

In case of adiabatic transformation, where we imagine to keep $\tilde{\mathcal{J}}$ (or ξ) fixed, while changing the temperature $b_i \to b_f$, we can measure the distance between initial and final states, and applying the general definition of the entropy of the Gibbs state, $S(b) = \log Z(b) - b\,\mathbb{E}_{\rho_b}[\tilde{\mathcal{J}}_H(\xi)]$, we get

$$\begin{aligned} S(b_f) - S(b_i) &\geq -b_f\,\mathbb{E}_{\rho_{b_f}}[\tilde{\mathcal{J}}_H(\xi)] + b_f\,\mathbb{E}_{\rho_{b_i}}[\tilde{\mathcal{J}}_H(\xi)] \\ &= -b_f\,\left(\mathbb{E}_{b_f}[\tilde{\mathcal{J}}_H(\xi)] - \mathbb{E}_{b_i}[\tilde{\mathcal{J}}_H(\xi)]\right) \\ &= -b_f\,\Delta Q \end{aligned}$$

By changing the global temperature, while keeping the reference observer/foliation associated with the generating vector field ξ, the entropy of the system increases, while the reservoir entropy decreases by an amount $b_f \Delta Q$, with an overall change in entropy

$$\Delta S + b_f \Delta Q \geq 0 \ .$$

This is the Clausius inequality of classical thermodynamics, which expresses the essential statement of the second law of thermodynamics.

Adiabatic Transformation

- A slow variation of the state realised via a series of equilibrium states defines a reversible *adiabatic* transformation, with

$$\Delta S = -b_f \Delta Q = b_f \left(\Delta \langle H^{(\varphi)}(\xi) \rangle + \Delta \langle P(\xi) \rangle \right) \, .$$

- The change in the global scaling $b_i \, \xi \to b_f \, \xi \in \mathrm{diff}(\mathcal{X})$ results in an increase of the entropy of the state of the fields.

- Differently form the isothermal process, here we do not perform a change on the embedding.

- In particular, the change in the entropy is not zero despite being measured along the same coadjoint orbit in $\mathfrak{g}^*$, a remarkable difference w.r.t. the same relation when derived in the formal setting of Lie group thermodynamics.

Thank You