Wrapped statistical models on $SE(n)$: motivation challenges and generalization to symmetrical spaces

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1. Random variables on $SE(n)$

- $X_1, \ldots, X_k$ i.i.d. R.V. $\Omega \rightarrow SE(n)$
- Goal: estimate the law from a set of samples $X_1, \ldots, X_k$ using a parametric model

Since $SE(n)$ has a bi-invariant Haar measure $d\nu$, we assume that the law of the $X_i$ has a density $f$ according to this measure.

2. Some classical probability densities on manifolds

(I) Heat kernels: Greens function of the heat equation $\frac{\partial f}{\partial \tau} = \Delta f$.
- Key ingredient: a Laplacian.

(II) If $h$ is a probability density on $\mathbb{R}^d$ depending on a Euclidean norm $\| \cdot \|$, define $f_\alpha(x)(d\nu) = \alpha h(\| \log_\alpha x \|) d\nu$ with $\alpha = \frac{1}{\int_M h(\| \log_\alpha x \|) d\nu}$.
- Key ingredients: a log map and a Euclidean distance in each tangent space.

(III) Wrapped distributions are defined by mapping a density $h$ on a tangent space $T_x M$ to the manifold using the exponential map: $f_\alpha(x) = (\exp \circ h)(x) = \frac{h(x)}{J_\alpha(x)}$ where $J_\alpha$ is the Jacobian determinant of the exponential map $\exp$.

3. Important characteristics of densities

- Expression of the density
  - (I) Heat kernels rarely admit explicit expressions on curved spaces.
  - (II) Can we compute the normalizing constant? Usually with Monte Carlo sampling.
  - (III) Can we compute the Jacobian $J_\alpha$?

- Moments
  - $\mathbb{E}[X]$ (Frechet means on Riemannian manifolds, bi-invariant means on Lie groups) are solution of
    \[ \int_M \log_\alpha(x) d\nu = E(\log_\alpha(x)) = 0 \]
  - When the mean is unique, we define the covariance as the vectorial covariance in the tangent space at the mean:
    \[ \Sigma = E(\log_\alpha(x) \otimes \log_\alpha(x)) \]
  - Given the density, can we easily obtain moments and vice-versa? (II) usually no / (III) if $J_\alpha$ is known then yes.
  - Given a statistical model of density are there estimators easily computable?

4. Probability densities on $SE(n)$

- We want to work with bi-invariant quantities: if the samples $X_i$ are composed with a rigid motion, we want the estimated law to be composed with the same motion.
- On Lie groups exponential map is equivariant under group multiplications and can hence be used to construct suitable probability densities.

As a manifold $SE(n)$ is a product between $SO(n)$ and $\mathbb{R}^n$ but the group structure is a semi-direct product:

\[ SE(n) = SO(n) \ltimes \mathbb{R}^n \]

and elements of its Lie algebra are parametrized by couples $(A, T)$ where $A$ is a skew-symmetric matrix and $T \in \mathbb{R}^n$. The differential of the exponential map on Lie groups at $u$ in the Lie algebra is given by the following formula:

\[ d\exp_u = d\exp_u \circ \left( \sum_{k \geq 2} \frac{(-1)^k}{(k+1)!} a_d^k \right) \]

which enables to compute the Jacobian determinant $J_\alpha$:

\[ J_\alpha = \det(d\exp_u) = \left( \prod_{i=0}^{n-1} 1 - \cos(\theta_i) \theta_i^\alpha \right) \times \ldots \times \left( \prod_{1 \leq i < j} \frac{1 + \cos(\theta_i + \theta_j)}{\theta_i + \theta_j} \frac{1 + \cos(\theta_i - \theta_j)}{\theta_i - \theta_j} \right) \]

where $\alpha = 1$ when $n$ is even and $2$ when $n$ is odd and $\theta_i$ are the angles of the planar rotations of the block diagonalization of $A$. This simplifies to $J_\alpha(T) = \left( \frac{2^{\frac{n-1}{2}} \cos(\theta_1)}{\|\theta\|^2} \right)$ on $SE(2)$.

Given a kernel $K$ on we can define probability distribution of type (III):

\[ f_{g,\Sigma}(\exp K(u)) = \frac{1}{J(u)/\sqrt{\det(\Sigma)}} K(\sqrt{\Sigma}^{-1}u) \]

If $\Sigma$ is sufficiently concentrated, $g$ and $\Sigma$ are the moments of $f_{g,\Sigma}$.

5. Density estimation on $SE(n)$

- Maximum likelihood estimator: no explicit expressions.
- Moment matching estimator: straightforward.

No law of large number or CLT on Lie groups $\rightarrow$ it is difficult to characterize convergence rates of moments $\rightarrow$ empirical comparison with the Euclidean case.

6. Generalization to symmetric spaces

- Can we find a natural definition of the logarithm?
- The Jacobian along a geodesic $\gamma$ follows a second order differential equation
  \[ J(t) = \det(A(t)) \]
  \[ A^\prime(t) + R(t)A(t) = 0 \]
  where
  \[ R(t) = R(\cdot, \gamma(t))\gamma(t) \]
  and $R$ is the curvature tensor of the connection. Since $\nabla R = 0$ on symmetric spaces $J$ should have an explicit expression.