Wrapped statistical models on $SE(n)$: motivation challenges and generalization to symmetric spaces

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ÉCOLE DE PHYSIQUE DES HOUCHES

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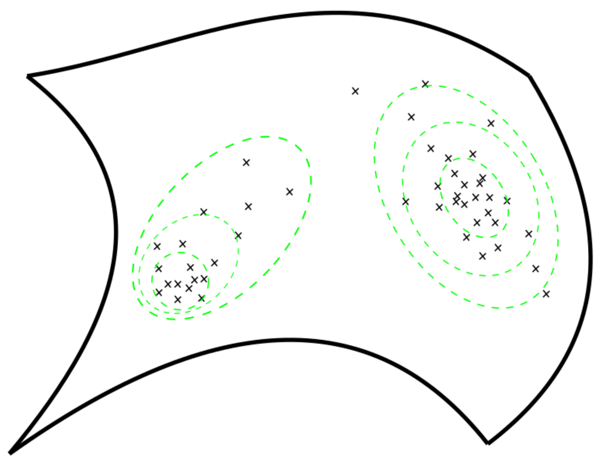
1. Random variables on $SE(n)$

- $X_1,..,X_k$ i.i.d. R.V. $\Omega \to SE(n)$
- Goal: estimate the law from a set of samples $x_1,..x_k$ using a parametric model



Since $SE(n)$ has a bi-invariant Haar measure dv we assume that the law of the X_i has a density f according to this measure.

2. Some classical probability densities on manifolds

(I) Heat kernels : Greens function of the heat equation $\frac{\partial f}{\partial f} = \Delta f$
◇ key ingredient : a Laplacian

(II) If h is a probability density on $\mathbb{R}^d$ depending on a Euclidean norm $\|.\|$, define

$$f_{\bar{x}}(x)dv = \alpha h(\|log_{\bar{x}}x\|)dv$$

with

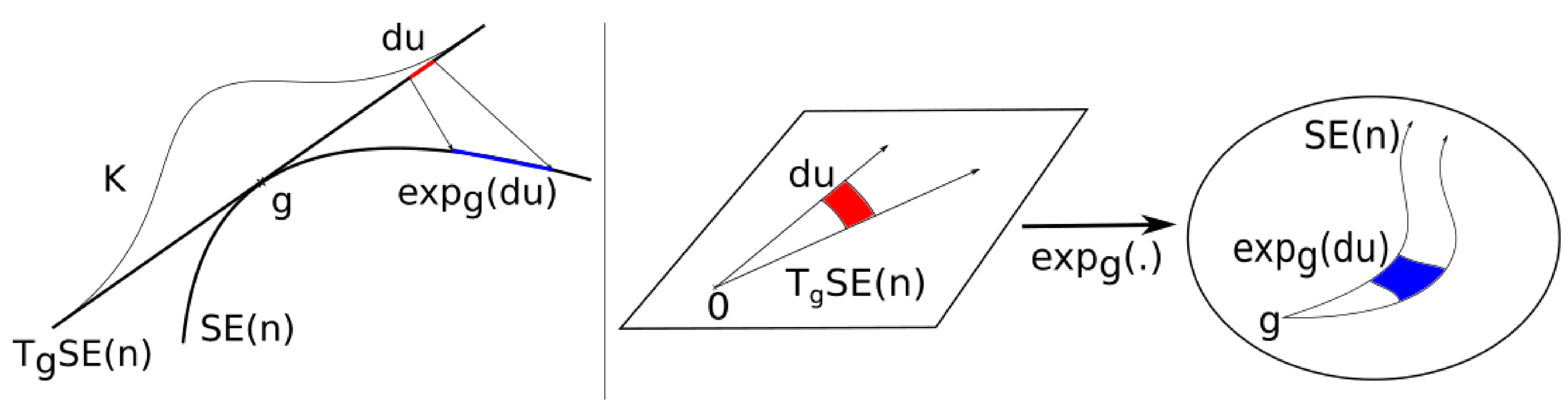
$$\alpha = \frac{1}{\int_{\mathcal{M}} h(\|log_{\bar{x}}x\|)dv}$$

◇ key ingredients: a **log** map and a Euclidean distance in each tangent spaces

(III) wrapped distributions are defined by mapping a density h on a tangent space $T_{\bar{x}}\mathcal{M}$ to the manifold using the exponential map:

$$f_{\bar{x}}(x) = (\exp_{\bar{x}\,*}h)(x) = \frac{h(x)}{J_{\bar{x}}(x)}$$

where J is the Jacobian determinant of the exponential



◇ key ingredients : **log** and **exp** maps

3. Important characteristics of densities

- Expression of the density
  (I) Heat kernels rarely admit explicit expressions on curved spaces
  (II) Can we compute the normalizing constant? Usually with Monte Carlo smapling
  (III) Can we compute the Jacobian J?
- Moments
  ▷ Means $\bar{x}$ (Frechet means on Riemannian manifolds, bi-invariant means on Lie groups) are solution of

$$\int_{\mathcal{M}} \log_{\bar{x}}(x)dv = E\left(\log_{\bar{x}}(x)\right) = 0$$

  ▷ When the mean is unique, we define the covariance as the vectorial covariance in the tangent space at the mean:

$$\Sigma = E\left(\log_{\bar{x}}(x) \otimes \log_{\bar{x}}(x)\right)$$

Given the density, can we easily obtain moments and vice-versa? (II) usually no / (III) if J is known then yes

- Given a statistical model of density are there estimators easily computable?

4. Probability densities on $SE(n)$

- We want to work with bi-invariant quantities: if the samples x_i are composed with a rigid motion, we want the estimated law to be composed with the same motion.
- On Lie groups exponential map is equivariant under group multiplications and can hence be used to construct suitable probability densities.

As a manifold $SE(n)$ is a product between $SO(n)$ and $\mathbb{R}^n$ but the group structure is a semi-direct product:

$$SE(n) = SO(n) \ltimes \mathbb{R}^n$$
$$(R,t)(R',t') = (RR', Rt' + t)$$

and elements of its Lie algebra are parametrized by couples (A,T) where A is a skew-symmetric matrix and $T \in \mathbb{R}^n$. The differential of the exponential map on Lie groups at u in the Lie algebra is given by the following formula:

$$d\exp_u = dL_{\exp_u} \circ \left(\sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} ad_u^k\right)$$

which enables to compute the Jacobian determinant J:

$$J = \det(d\exp_u) = \left(\prod_i 2\frac{1-cos(\theta_i)}{\theta_i^2}\right)^{\alpha} \times ..$$
$$..\prod_{i<j}\left(4.\frac{1+cos(\theta_i+\theta_j)}{(\theta_i+\theta_j)^2}.\frac{1+cos(\theta_i-\theta_j)}{(\theta_i-\theta_j)^2}\right)$$

where $\alpha = 1$ when n is even and 2 when n is odd and θ_i are the angles of the planar rotations of the block diagonalization of A. This simplifies to $J(\theta, T) = \left|2\frac{1-cos(\theta)}{\theta^2}\right|$ on $SE(2)$.
Given a kernel K on we can define probability distribution of type (III):

$$f_{g,\Sigma}(\exp_g(u)) = \frac{1}{J(u)\sqrt{\det(\Sigma)}} K\left(\sqrt{u^t\Sigma^{-1}u}\right)$$

If Σ is sufficiently concentrated, g and Σ are the moments of $f_{g,\Sigma}$.

5. Density estimation on $SE(n)$

- Maximum likelihood estimator: no explicit expressions
- moment matching estimator: straightforward
  No law of larges number or CLT on Lie groups $\to$ it is difficult to characterize convergence rates of moments $\to$ empirical comparison with the Euclidean case:

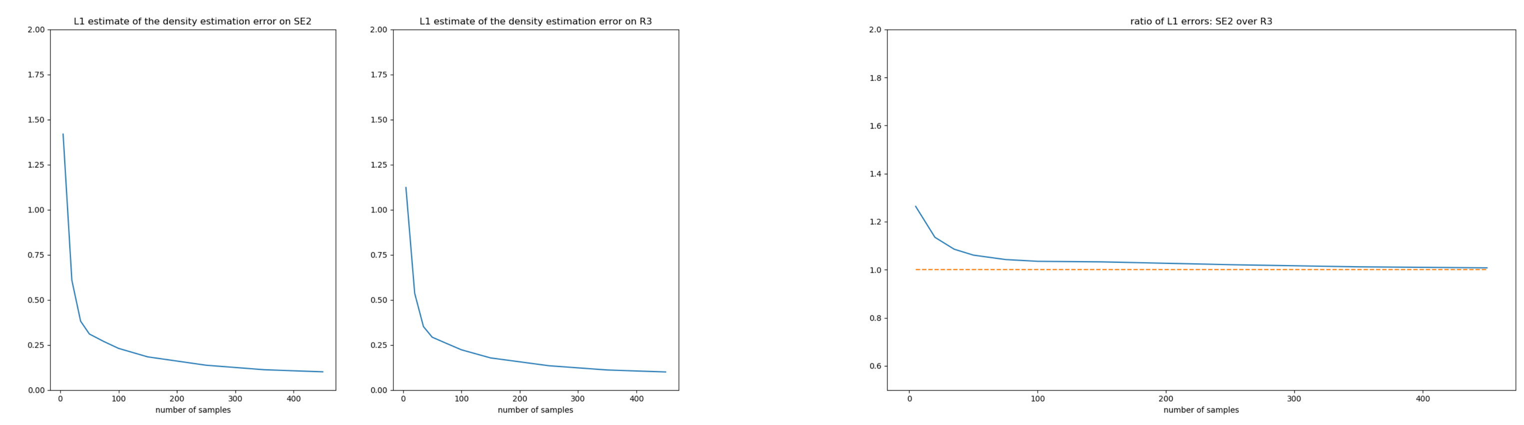


Figure: L^1 errors and their ratios on $SE(2)$ and $T_eSE(2) \sim \mathbb{R}^3$ for an anisotropic covariance Σ.

7. Generalization to symmetric spaces

- Can we find a natural definition of the logarithm?
- The Jacobian along a geodesic γ follows a second order differential equation

$$J(t) = \det(A(t))$$
$$A''(t) + R(t)A(t) = 0$$

where

$$R(t)(.) = \mathbf{R}(.,\gamma'(t))\gamma'(t)$$

and $\mathbf{R}$ is the curvature tensor of the connection. Since $\nabla R = 0$ on symmetric spaces J should have an explicit expression.

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