Viscoelastic flows of Maxwell fluids with conservation laws

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Abstract

Maxwell introduced a 1D *causal* relaxation model for viscoelastic fluid flows, where information propagates at *finite* speed. Most viscoelastic models for multi-dimensional flows (Oldroyd-B etc.) add diffusion and lack the *local* character of Maxwell model ! Here, a *symmetric-hyperbolic system of conservation laws* defines multi-D viscoelastic compressible flows as extensions of polyconvex elastodynamics. In the shallow-water regime, the system reduces to a useful viscoelastic extension of Saint-Venant 2D model.

Elastic and viscous motions in continuum mechanics

- Bodies $\mathcal{B}$ are Riemannian manifolds with metric $G_{\alpha\beta} \in S^{++}(\mathbb{R}^{d\times d})$, $d=3$ in coordinates $\{a^\alpha\}$; flows are configurations $\phi_t(\mathcal{B})$, $t\in\mathbb{R}$ into the Euclidean ambiant space with a given force field f, $f^i e_i$ in coordinates $\{x^i\}$
- Galilean-invariant balance of total energy $E \geq 0$ holds given heat supply R

$$\partial_t(E\circ\phi_t) = \mathrm{div}_a(S^{i\alpha}\partial_t\phi_t^i) + \partial_t\phi_t^i(f^i\circ\phi_t) + R \tag{1}$$

- Momentum balance holds with mass-density $\hat\rho(a)\geq 0$

$$\hat\rho(\partial^2_{tt}\phi_t) = \mathrm{div}_a\, S + \hat\rho(f\circ\phi_t) \tag{2}$$

given $r\circ\phi_t = R/\hat\rho$ and internal energy $e\circ\phi_t := \frac{1}{\hat\rho}E\circ\phi_t - \frac12|\partial_t\phi_t|^2$

$$\hat\rho(\partial_t e\circ\phi_t) - S^{i\alpha}\partial^2_{t\alpha}\phi_t^i = \hat\rho(r\circ\phi_t) \tag{3}$$

where Piola-Kirchoff stress S satisfies $S^{i\alpha}\partial_\alpha\phi^j = S^{j\alpha}\partial_\alpha\phi^i$.

- Smooth motions with $F^i_\alpha := \partial_\alpha\phi^i_t\circ\phi_t^{-1}$, $u^i := \partial_t\phi^i_t\circ\phi_t^{-1}$ are defined by, for e.g. *Neo-Hookean elastic* (solid) bodies $e(F^k_\alpha F^k_\alpha) := \frac{\mu}{2}(F^k_\alpha F^k_\alpha - d)$,

$$\begin{aligned}
&\partial_t\left(\hat\rho\, u^i\circ\phi_t\right) - \partial_\alpha S^{i\alpha} = \hat\rho f^i\\
&\partial_t\left(F^i_\alpha\circ\phi_t\right) - \partial_\alpha\left(u^i\circ\phi_t\right) = 0\\
&\partial_t\left(|F^i_\alpha|\circ\phi_t\right) - \partial_\alpha\left(C^\alpha_j\circ\phi_t\, u^j\circ\phi_t\right) = 0\\
&\partial_t\left(C^\alpha_i\circ\phi_t\right) + \sigma_{ijk}\sigma_{\alpha\beta\gamma}\partial_\beta\left(F^j_\gamma\circ\phi_t\, u^k\circ\phi_t\right) = 0
\end{aligned} \tag{4}$$

with $S^{i\alpha} = \hat\rho\partial_{F^i_\alpha}e$, $C^\alpha_i = \sigma_{ijk}\sigma_{\alpha\beta\gamma}F^j_\beta F^k_\gamma$ and Levi-Civita symbol σ_{ijk}, or

$$\begin{aligned}
&\partial_t\left(\rho u^i\right) + \partial_j\left(\rho u^j u^i - \sigma^{ij}\right) = \rho f^i\\
&\partial_t\left(\rho F^i_\alpha\right) + \partial_j\left(\rho u^j F^i_\alpha - \rho F^j_\alpha u^i\right) = 0\\
&\partial_t\rho + \partial_j\left(\rho u^j\right) = 0\\
&\partial_t\left(\rho C^\alpha_i\right) + \partial_j\left(\rho u^j C^\alpha_i\right) + \sigma_{ijk}\sigma_{\alpha\beta\gamma}\partial_l\left(|F^i_\alpha|^{-1}F^l_\beta F^j_\gamma u^k\right) = 0
\end{aligned} \tag{5}$$

with Cauchy stress $\sigma^{ij} := |F^i_\alpha|^{-1}F^j_\alpha S^{i\alpha}\circ\phi_t^{-1}$ and $\rho := |F^i_\alpha|^{-1}\hat\rho$.

$$\text{Recall Piola's identities:}\quad \partial_j(|F^i_\alpha|^{-1}F^j_\alpha) = 0 \quad \forall\, i = 1\dots d. \tag{6}$$

- *Polytropic* fluids $e(\rho) := \frac{C_0}{\gamma-1}\rho^{\gamma-1}$, $\sigma^{ij} = (-C_0\rho^\gamma)\delta^{ij} + \tau^{ij}$ can be viscous

$$\tau^{ij} = 2\dot\mu D(u)^{ij} + \ell\, D(u)^{kk}\,\delta_{ij} \tag{7}$$

insofar as τ^{ij} is objective and $\partial_t\eta + (u^j\partial_j)\eta := \tau^{ij}D(u)^{ij}/\theta \geq 0$

- Instead of entropy η one often uses temperature $\theta = -\partial_\eta e$ and $\psi = e - \theta\eta$

$$\hat\rho\left((\eta\circ\phi_t)\partial_t(\theta\circ\phi_t) + \partial_t(\psi\circ\phi_t)\right) - S^{i\alpha}\partial_\alpha(u^i\circ\phi_t) = -\hat\rho D\circ\phi_t \tag{8}$$

with $S^{i\alpha} = \hat\rho\partial_{F^i_\alpha}\psi$, $\eta = -\partial_\theta\psi$ and a dissipation $D\geq 0$

Viscoelastic flows by Maxwell fluids

- Assume $\psi = \frac{C_0(\theta)}{\gamma-1}\rho^{\gamma-1} + \mathcal{F}(c,\theta)$ with $c \approx \mathbb{E}(R\otimes R)$ given by

$$dR^i = \left(-(u^j\partial_j)R^i + (\partial_j u^i)R^j - \frac{2K}{\xi}F^i(R)\right)dt + \sqrt{\frac{4k_B\theta}{\xi}}dW^i(t) \tag{9}$$

where ξ denotes friction, and $K(\theta)$ a spring factor for $F^i(R) = \mathcal{H}'R^i$

- If $\mathcal{H}'$ is constant, then R is Gaussian and it exactly holds

$$\overset{\diamond}{c}{}^{ij} = -\frac{4K\mathcal{H}'}{\xi}c^{ij} + \frac{4k_B\theta}{\xi}\delta^{ij} \tag{10}$$

with $\overset{\diamond}{\tau}{}^{ij} = \partial_t\tau^{ij} + u^k\partial_k\tau^{ij} - \partial_k u^i\tau^{kj} - \tau^{ik}\partial_k u^j$. A dissipative choice is

$$\mathcal{F} = K\mathcal{H}\left(tr(c)\right) - k_B\theta\log|c|\,, \quad \tau^{ij} = 2\rho(K\mathcal{H}'c^{ij} - k_B\theta\delta^{ij})\,. \tag{11}$$

It produces the *Upper-Convected Maxwell* (UCM) rheological equation

$$\lambda\,\overset{\diamond}{\tau} + \mathrm{div}\,u\,\lambda\,\tau + \tau = 2\dot\mu D(u) \tag{12}$$

with $\lambda = 4K\mathcal{H}'/\zeta$, $\dot\mu/\lambda = \mu = 2\rho k_B\theta$, & well-defined motions... if *1D* !

- Now, in smooth motions, $A^{\alpha\beta} = [F^{-1}]^\alpha_i c^{ij}[F^{-1}]^\beta_j$ satisfies

$$\partial_t\left(A^{\alpha\beta}\circ\phi_t\right) = \frac{4k_B\theta}{\xi}\left([F^{-1}\circ\phi_t]^\alpha_i[F^{-1}\circ\phi_t]^\beta_i\right) - \frac{4K\mathcal{H}'}{\xi}(A^{\alpha\beta}\circ\phi_t) \tag{13}$$

- K-BKZ fluids have well-defined motions by an *integro-differential* system using

$$c^{ij}\circ\phi_t = \frac{k_B\theta}{K\mathcal{H}'}\int_{t_0}^t ds\,\frac{1}{\lambda}e^{\frac{s-t}{\lambda}}(F^i_\alpha\circ\phi_t[F^{-1}]^\alpha_k\circ\phi_s)([F^{-1}]^\beta_k\circ\phi_s F^j_\beta\circ\phi_t) \tag{14}$$

- Here, we propose the following *system of conservation laws* for UCM fluids:

Isothermal viscoelastic flows of compressible UCM fluids

$$\begin{aligned}
&\partial_t(\rho u^i) + \partial_j\left(\rho u^j u^i + (p + 2\rho k_B\theta)\delta^{ij} - 2\rho K\mathcal{H}'F^i_\alpha A^{\alpha\beta}F^j_\beta\right) = \rho f^i\\
&\partial_t(\rho F^i_\alpha) + \partial_j\left(\rho u^j F^i_\alpha - \rho u^i F^j_\alpha\right) = 0\\
&\partial_t\rho + \partial_i(u^i\rho) = 0\\
&\partial_t(\rho A^{\alpha\beta}) + \partial_j\left(\rho u^j A^{\alpha\beta}\right) = \frac{4\rho}{\xi}\left(k_B\theta\left([F^{-1}]^\alpha_i[F^{-1}]^\beta_i\right) - K\mathcal{H}'A^{\alpha\beta}\right)
\end{aligned} \tag{15}$$

Theorem: well-posed Cauchy problems on small times

The system of conservation laws (15) admits univoque classical solutions $C^1([0,T)\times\mathbb{R}^d)$ in $\mathcal{A}^+ := \{\rho > 0,\ A = A^T > 0\}$ for small enough $T > 0$.

Proof Cauchy problems are well-posed on small times for systems of conservation laws that are symmetric-hyperbolic, with a *strictly* convex extension (i.e. with an additional conservation law for a "mathematical entropy" strictly convex in conservative variables).

Application to free-surface gravity flows in shallow-water regime

- Hydraulics often considers flows under $(f^x, f^y, f^z) := (0,0,-g)$ of fluids filling $\mathcal{D}_t := \{z^b(x,y) < z < z^b(x,y) + H(t,x,y)\}$ with $0 \leq H \ll L$ given by

$$\partial_t H + u^x\partial_x(z^b + H) + u^y\partial_y(z^b+H) = u^z\sqrt{1 + |\nabla_H(z^b+H)|^2} \tag{16}$$

$\nabla_H = (\partial_x,\partial_y)$, & with $\nabla_H z^b \ll H/L$, $\tau^{xz}_\epsilon, \tau^{yz}_\epsilon \ll \rho_0 g L$ given $L > 0$, $\rho\equiv\rho_0$.

- The free-surface flows (16)–(5) with small Navier friction k at boundaries and $U^i := \frac1H\int_{z^b}^{z^b+H}u^i = u^i + O(H/L)^2$, $i = x,y$ are approximately 2D. One uses

$$\partial_t H + \mathrm{div}_H(HU) = 0 \tag{17}$$

$$\partial_t(HU) + \mathrm{div}_H\left(HU\otimes U - H\Sigma\right) = -gH\nabla_H(z_b+H) - kHU \tag{18}$$

with e.g. $\Sigma := \frac{1}{\rho_0}\frac1H\int_{z^b}^{z^b+H}dz\,(\tau^{ij} - \tau^{zz}) = \nu\left(\nabla_H U + \nabla_H U^T - 2I\,\mathrm{div}_H U\right) \ll gL$.

- Using (15), Saint-Venant system (17)–(18) can be extended to viscoelastic flows:

$$\begin{aligned}
&\Sigma = \mathcal{G}H\left(F^H A^H (F^H)^T - (A^{cc}H^2)I\right) \quad H = \hat H|F^H|^{-1}, \hat H > 0\\
&\partial_t(HF^H) + \mathrm{div}_H(HU\otimes F^H - HF^H\otimes U) = 0\\
&\partial_t(HA^H) + \mathrm{div}_H(HUA^H) = H\left(((F^H)^T F^H)^{-1} - A^H\right)/\lambda\\
&\partial_t(HA^{cc}) + \mathrm{div}_H(HUA^{cc}) = H\left(H^{-2} - A^{cc}\right)/\lambda
\end{aligned} \tag{19}$$

- The system of conservation laws (17)–(18)–(19) is symmetric-hyperbolic in $\mathcal{A}^+_H := \{\rho > 0,\ A^H = (A^H)^T > 0\}$. Given smooth initial values at $t = 0$, it has univoque classical solutions in $C^1([0,T)\times\mathbb{R}^d)$ for small $T > 0$, equivalently defined in Lagrangian description by $U\circ\Phi^H_t = \partial_t\Phi^H_t$, $F^H\circ\Phi^H_t = \nabla_H\Phi^H_t$.
- Viscoelastic shear *wave* flows like $\Phi^H_t(a) = a + X(t,b)e_a$, $X(t\leq 0, b) = 0$, $X(t,0) = \Delta X.1_{t>0}$ for $b > 0$ sol. of (20) can be considered beyond 1D!

$$\partial^2_{tt}X(t,b) - \mathcal{G}_\epsilon\partial^2_{bb}X(t,b) = \frac{\mathcal{G}_\epsilon}{\lambda}\int_0^t ds\; e^{\frac{t-s}{\lambda}}\partial^2_{bb}X(s,b) \quad t, b > 0 \tag{20}$$

i.e., since $\partial^2_{bb}X(t,b) = 0 = \partial_t X(t,b)$, $t\leq 0$:

$$\omega\hat X = \frac{\mathcal{G}_\epsilon}{\omega + \frac1\lambda}\partial^2_{bb}\hat X$$

is solved using $\hat X(\omega,b) = \int_0^\infty dt\; e^{-\omega t}X(t,b)$:

$$\frac{X}{\Delta X} = \left(e^{-y} + y\int_y^{\frac{t}{\lambda}}dr\frac{I_1\left(\sqrt{r^2-y^2}\right)}{e^r\sqrt{r^2-y^2}}\right)1_{t>\frac{b}{\sqrt{\mathcal{G}_\epsilon}}}$$

and numerically computed in Fig. 1

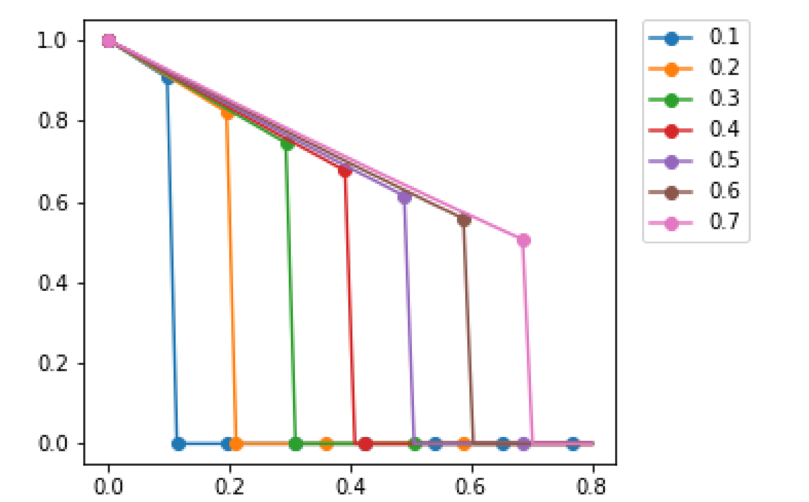

Figure 1: 1D shear wave $X/\Delta X$ function of $y = b/\lambda\sqrt{\mathcal{G}_\epsilon}$ at $t/\lambda \in \{.1, .2\dots.7\}$.

Conclusion

- *Multi-dimensional* (smooth) viscoelastic flows of Maxwell fluids are now well-defined by conservation laws, (15) or (17)–(18)–(19).
- Key to the new viscoelastic systems are *material variables* added into polyconvex elastodynamics (for hyperelastic continuum), here to model *viscous* inelasticities.
- The framework reaches Navier-Stokes after relaxation, and covers many rheologies.

All preprints (and **references**, therein) are on http://hal.archives-ouvertes.fr/.