THALES

SPIGL'20

ÉCOLE DE PHYSIQUE DES HOUCHES

Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning

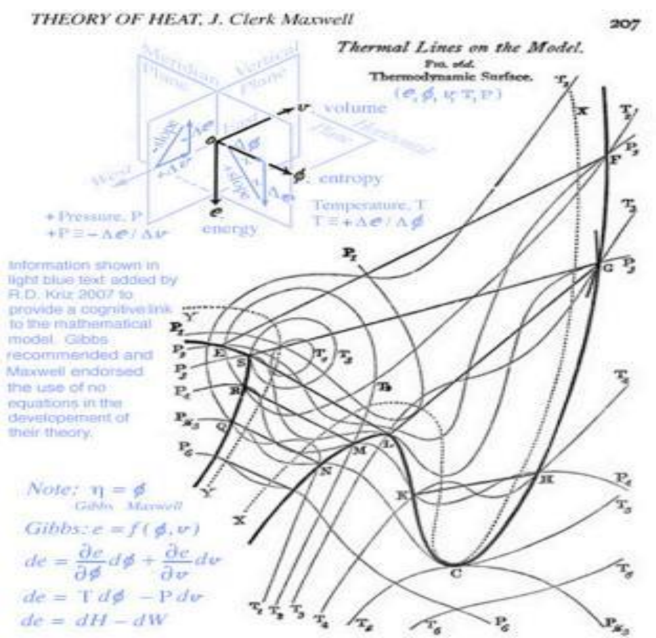
Souriau-Casimir Lie Groups Thermodynamics & Machine Learning

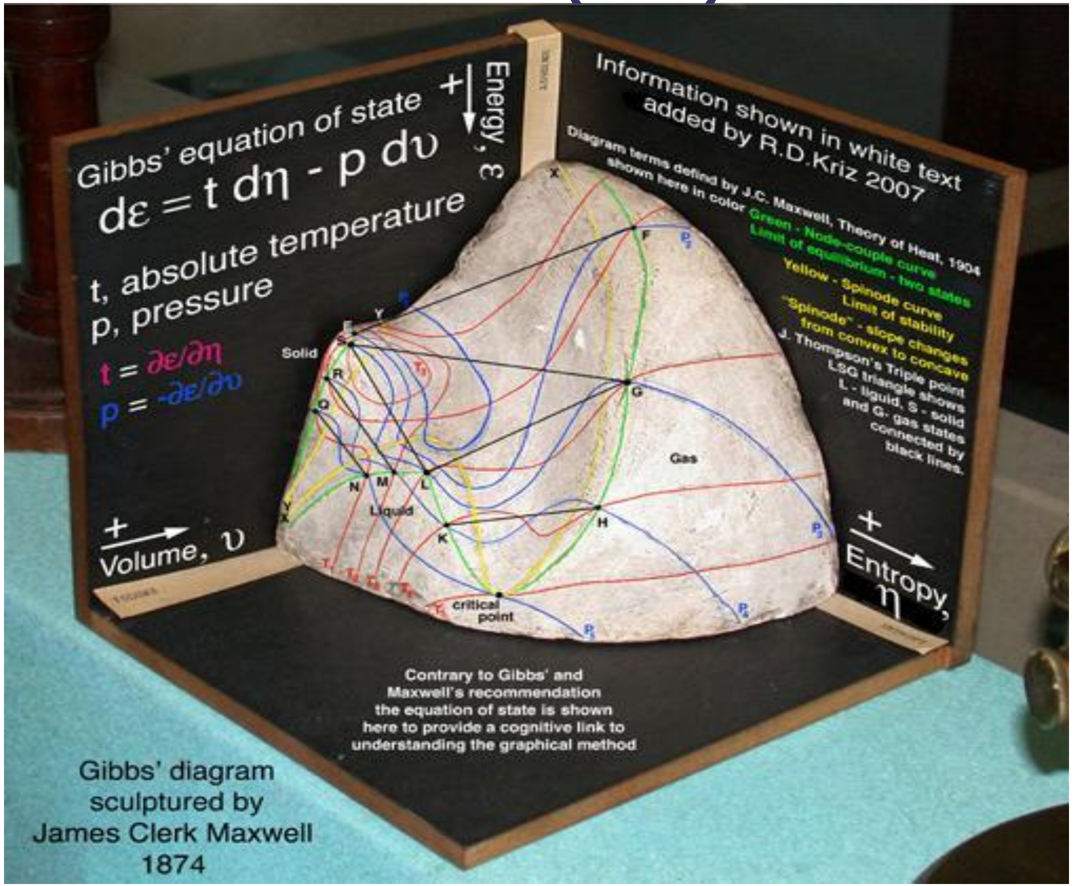
Frédéric BARBARESCO

28/07/2020

« Tout mathématicien sait qu'il est impossible de comprendre un cours élémentaire en thermodynamique. »
Vladimir Arnold

Geometric Theory of Heat:
Gibbs Diagrams sculpted by James Clerk Maxwell (1874)





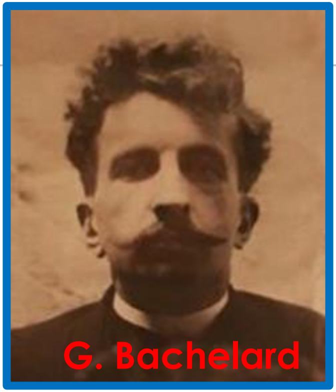
Jean-Marie Souriau (1922-2012): New way of thinking Physics

« Il est évident que l'on ne peut définir de valeurs moyennes que sur des objets appartenant à un espace vectoriel (ou affine); donc - si bourbakiste que puisse sembler cette affirmation - que l'on n'observera et ne mesurera de valeurs moyennes que sur des grandeurs appartenant à un ensemble possédant physiquement une structure affine. Il est clair que cette structure est nécessairement unique - sinon les valeurs moyennes ne seraient pas bien définies. » -
Jean-Marie Souriau

« Il n'y a rien de plus dans les théories physiques que les groupes de symétrie si ce n'est la construction mathématique qui permet précisément de montrer qu'il n'y a rien de plus » -
Jean-Marie Souriau
[There is nothing more in physical theories than symmetry groups except the mathematical construction which allows precisely to show that there is nothing more]

Gaston Bachelard – Le nouvel esprit scientifique

« La Physique mathématique, en incorporant à sa base la notion de groupe, marque la suprématie rationnelle… Chaque géométrie – et sans doute plus généralement chaque organisation mathématique de l'expérience – est caractérisée par un groupe spécial de transformations…. Le groupe apporte la preuve d'une mathématique fermée sur elle-même. Sa découverte clôt l'ère des conventions, plus ou moins indépendantes, plus ou moins cohérentes » - Gaston Bachelard, Le nouvel esprit scientifique, 1934



« Sous cette aspiration, la physique qui était d'abord une science des "agents" doit devenir une science des "milieux". C'est en s'adressant à des milieux nouveaux que l'on peut espérer pousser la diversification et l'analyse des phénomènes jusqu'à en provoquer la géométrisation fine et complexe, vraiment intrinsèque… Sans doute, la réalité ne nous a pas encore livré tous ses modèles, mais nous savons déjà qu'elle ne peut en posséder un plus grand nombre que celui qui lui est assigné par la théorie mathématique des groupes. » – Gaston Bachelard, Etude sur l'Evolution d'un problème de Physique
La propagation thermique dans les solides, 1928

http://www.vrin.fr/book.php?title_url=Etude_sur_l_evolution_d_un_probleme_de_physique_La_propagation_thermique_dans_les_solides_9782711600434&search_back=&editor_back=%&page=2

Jean-Marie Souriau Seminal Paper - 1974

Statistical Mechanics, Lie Group and Cosmology - 1st part: Symplectic Model of Statistical Mechanics

Jean-Marie Souriau

Abstract: The classical notion of Gibbs' canonical ensemble is extended to the case of a symplectic manifold on which a Lie group has a symplectic action ("dynamic group"). The rigorous definition given here makes it possible to extend a certain number of classical thermodynamic properties (temperature is here an element of the Lie algebra of the group, heat an element of its dual), notably inequalities of convexity. In the case of non-commutative groups, particular properties appear: the symmetry is spontaneously broken, certain relations of cohomological type are verified in the Lie algebra of the group. Various applications are considered (rotating bodies, covariant or relativistic statistical Mechanics). [These results specify and complement a study published in an earlier work (*), which will be designated by the initials SSD].

(*) Souriau, J.-M., Structure des systèmes dynamique. Dunod, collection Dunod Université, Paris 1969. http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Souriau, J-M., Mécanique statistique, groupes de Lie et cosmologie, Colloques Internationaux C.N.R.S., n°237 – Géométrie symplectique et physique mathématique, pp.59-113, 1974

English translation by F. Barbaresco:

https://www.academia.edu/42630654/Statistical_Mechanics_Lie_Group_and_Cosmology_1_st_part_Symplectic_Model_of_Statistical_Mechanics

Souriau SSD Chapter IV: Gibbs Equilibrium is not covariant with respect to Dynamic Groups of Physics

MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais *ce n'est pas un sous-groupe invariant*, ainsi que le montre un calcul trivial. Si un système dynamique est *conservatif* dans un repère d'inertie, il en résulte qu'il peut *ne plus être conservatif dans un autre*. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.77) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G'.

Soit $\mathcal{G}'$ l'algèbre de Lie G' ; on sait que $\mathcal{G}'$ est une sous-algèbre de Lie de celle de G, notée $\mathcal{G}$; un équilibre du système sera caractérisé par un élément Z de $\mathcal{G}'$, donc de $\mathcal{G}$; on pourra écrire

(17.78)
$$Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de Z est associé un élément M du dual $\mathcal{G}'^*$ de $\mathcal{G}'$, valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28).

(17.79) On voit que c'est Z (17.78) qui généralise la « température » ; le théorème d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre *correspondant à la même valeur de* Z ; l'entropie s, le potentiel de Planck z et le moment moyen M sont *additifs*. W

J.M. Souriau, Structure des systèmes dynamiques, Chapitre IV « Mécanique Statistique »

Trompette de Souriau

Lorsque le fait qu'on rencontre est en opposition avec une théorie régnante, il faut accepter le fait et abandonner la théorie, alors même que celle-ci, soutenue par de grands noms, est généralement adoptée

- Claude Bernard "Introduction à l'Étude de la Médecine Expérimentale"

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Main references for Souriau « Lie Groups Thermodynamics »

1966

SUPPLEMENTO AL NUOVO CIMEN
VOLUME IV

N. 1, 1966

Définition covariante des équilibres thermodynamiques.

J.-M. SOURIAU
Faculté des Sciences - Marseille

(ricevuto il 5 Novembre 1965)

CONTENTS. — 1. Un problème variationnel. – 2. Mécanique statistique classique. – 3. Equilibres permis par un groupe de Lie. – 4. Exemples. – 5. Localisation de la température vectorielle.

1974

Colloques Internationaux C.N.R.S.
N° 237 – Géométrie symplectique et physique mathématique

MÉCANIQUE STATISTIQUE, GROUPES DE LIE ET COSMOLOGIE

Jean-Marie SOURIAU (1)

Première partie
FORMULATION SYMPLECTIQUE DE LA MECANIQUE STATISTIQUE

Référence to Blanc-Lapierre Book in Souriau Book

[7] A. Blanc-Lapierre, P. Casal, and A. Tortrat, *Méthodes mathématiques de la mécanique statistique*, Masson, Paris, 1959.

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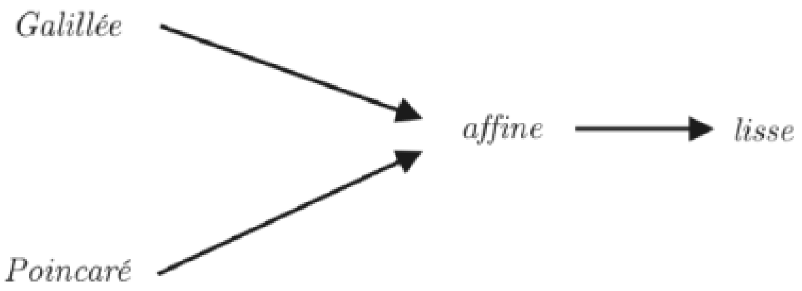
Souriau Quinta Essentia (Quinte Essence)

- *"Il y a un théorème qui remonte au XXème siècle. Si on prend une orbite coadjointe d'un groupe de Lie, elle est pourvue d'une structure symplectique. Voici un algorithme pour produire des variétés symplectiques : prendre des orbites coadjointes d'un groupe. Donc cela laisse penser que derrière cette structure symplectique de Lagrange, il y avait un groupe caché. Prenons le mouvement classique d'un moment du groupe, alors ce groupe est très «gros» pour avoir tout le système solaire. Mais dans ce groupe est inclus le groupe de Galilée, et tout moment d'un groupe engendre des moments d'un sous-groupe. On va retrouver comme cela les moments du groupe de Galilée, et si on veut de la mécanique relativiste, cela va être celui du groupe de Poincaré. En fait avec le groupe de Galilée, il y a un petit problème, ce ne sont pas les moments du groupe de Galilée qu'on utilise, ce sont les moments d'une extension centrale du groupe de Galilée, qui s'appelle le groupe de Bargmann, et qui est de dimension 11. C'est à cause de cette extension, qu'il y a cette fameuse constante arbitraire figurant dans l'énergie. Par contre quand on fait de la relativité restreinte, on prend le groupe de Poincaré et il n'y a plus de problèmes car parmi les moments il y a la masse et l'énergie c'est mc^2. Donc le groupe de dimension 11 est un artéfact qui disparait, quand on fait de la relativité restreinte."*

SOURIAU: Affine Group and Thermodynamics

- « Les différentes versions de la science mécanique peuvent se classer par la géométrie que chacune implique pour l'espace et le temps ; géométrie qui se détermine par le groupe de covariance de la théorie. Ainsi la mécanique newtonienne est covariante par le **groupe de Galilée**; la relativité restreinte par le **groupe de Lorentz-Poincaré** ; la relativité générale par le **groupe « lisse »** (le groupe des difféomorphismes de l'espace-temps). Il existe cependant une partie des énoncés de la mécanique dont la covariance appartient à un quatrième groupe – rarement envisagé : **le groupe affine**. Groupe qui figure dans le diagramme d'inclusion suivant :



- Comment se fait-il qu'un point de vue unitaire, (**qui serait nécessairement une véritable Thermodynamique**), ne soit pas encore venu couronner le tableau ? Mystère... »

Fundamental Equation of Geometric Thermodynamic: Entropy Function is an Invariant Casimir Function in Coadjoint Representation

Entropy S

$$S:\mathfrak{g}^* \to R$$
$$Q \mapsto S(Q)$$

Heat Q, (Planck) Temperature β and Φ Massieu Characteristic Function

$$S(Q)=\langle \beta, Q\rangle - \Phi(\beta)\ ,\ Q=\frac{\partial \Phi(\beta)}{\partial \beta}\in \mathfrak{g}^*\ \ ,\ \beta=\frac{\partial S(Q)}{\partial Q}\in \mathfrak{g}$$

Invariance of Entropy S Under the action of the Group

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right)=S(Q)$$

New Definition of Entropy S as Invariant Casimir Function in Coadjoint Representation

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

Moment Map J

Lie Groups Thermodynamic Equations and its extension (1/3)

Q : Heat, element of dual Lie Algebra

β : (Planck) température element of Lie algebra

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

$\theta(g)$: Souriau Cocyle

Φ : Massieu Characteristic Function

J : Souriau Moment Map

$$\Phi(\beta) = -\log \int_M e^{-\langle \beta, J(\xi) \rangle} d\lambda$$

$$J : M \to \mathfrak{g}^*$$

S : Entropy

Legendre Transform

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

I : Fisher Information Metric

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2}$$

Lie Groups Thermodynamic Equations and its extension (2/3)

Entropy Invariance under the action of the Group !

$$S\left(Ad_g^{\#}(Q)\right)=S(Q)$$

$$Ad_g^{\#}(Q)=Ad_g^{*}(Q)+\theta(g)$$

Entropy Solution of Casimir Equation

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau cocycle

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\theta(g)=Q\left(Ad_g(\beta)\right)-Ad_g^{*}(Q)$$

Souriau characteristic of the foliation

$$\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$$

Entropy & Poisson Bracket →

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

Lie Groups Thermodynamic Equations and its extension (3/3)

Entropy Production

$$dS=\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)dt$$

2nd principle is related to positivity of Fisher tensor

$$\frac{dS}{dt}=\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)\geq 0$$

Metric Tensor related to Fisher Metric

$$\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)=\tilde{\Theta}\left(\frac{\partial H}{\partial Q},\beta\right)+\left\langle Q,\left[\frac{\partial H}{\partial Q},\beta\right]\right\rangle$$

Time Evolution of Heat wrt to Hamiltonian H

$$\frac{dQ}{dt}=\{Q,H\}_{\tilde{\Theta}}=ad^{*}_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)$$

Stochastic Equation

$$dQ+\left[ad^{*}_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)\right]dt+\sum_{i=1}^{N}\left[ad^{*}_{\frac{\partial H_i}{\partial Q}}Q+\Theta\left(\frac{\partial H_i}{\partial Q}\right)\right]\circ dW_i(t)=0$$

Euler-Poincaré Equation in case of Non-Null Cohomology

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$Q = \frac{\partial \Phi}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \beta} = ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta} + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$\left(ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta}\right)_j + \Theta\left(\frac{\partial H}{\partial Q}\right)_j = C^k_{ij}\, ad^*_{\left(\frac{\partial H}{\partial Q}\right)^i} \left(\frac{\partial \Phi}{\partial \beta}\right)_k + \Theta_j$$

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître » - Henri Poincaré, CRAS, 18 Février 1901

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique*. Note de M. H. POINCARÉ.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.$$

« Elles sont surtout intéressantes dans le cas où U étant nul, T ne dépend que des η » - Henri Poincaré

de Saxcé, G. Euler-Poincaré equation for Lie groups with non null symplectic cohomology. Application to the mechanics. In GSI 2019. LNCS; Nielsen, F., Barbaresco, F., Eds.; Springer: Berlin, Germany, 2019; Volume 11712

Souriau Model Variational Principle : Poincaré-Cartan Integral Invariant on Massieu Characteristic Function

Extension of Poincaré-Cartan Integral Invariant for Souriau Model

$$\omega = \langle Q, (\beta.dt) \rangle - S.dt = \left(\langle Q, \beta \rangle - S \right).dt = \Phi(\beta).dt$$

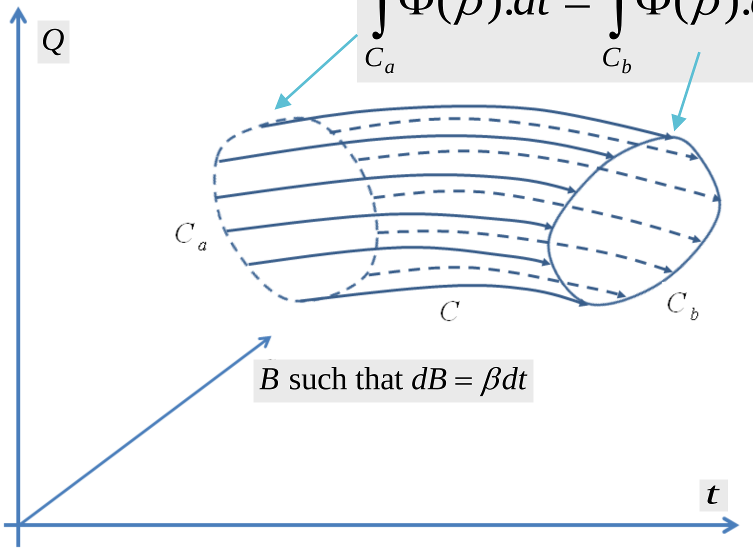
$$g(t) \in G \qquad \beta(t) = g(t)^{-1} \dot{g}(t) \in \mathfrak{g}$$

Variational Model for arbitrary path $\eta(t)$

$$\delta\beta = \dot{\eta} + [\beta, \eta]$$

$$\delta \int_{t_0}^{t_1} \Phi(\beta(t)).dt = 0$$

$$\int_{C_a} \Phi(\beta).dt = \int_{C_b} \Phi(\beta).dt$$



Legendre Transform as Reciprocal Polar with respect to a paraboloid

$$\Phi(\beta) = \langle \beta, Q \rangle - S(Q)$$

$\Phi(\beta)$ Reciprocal Polar with respect to the Paraboloid $Q^2 = 2S(Q)$

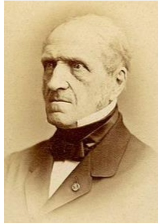
Darboux Lecture on Legendre Transform based on Chasles remark

178. La méthode de Legendre est élégante et irréprochable. Elle consiste à remplacer les variables x, y, z par p, q et

$$v = px + qy - z,$$

en considérant v comme une fonction de p et de q ; ce qui revient, suivant une remarque de Chasles, à substituer à la surface sa polaire réciproque par rapport au paraboloïde ayant pour équation

$$2z = x^2 + y^2.$$




Koszul Book on Souriau Work: The Little Green Book

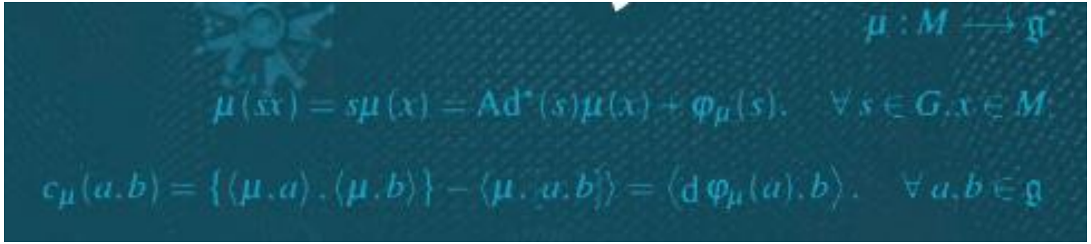
Koszul Book on Souriau Work: The Little Green Book

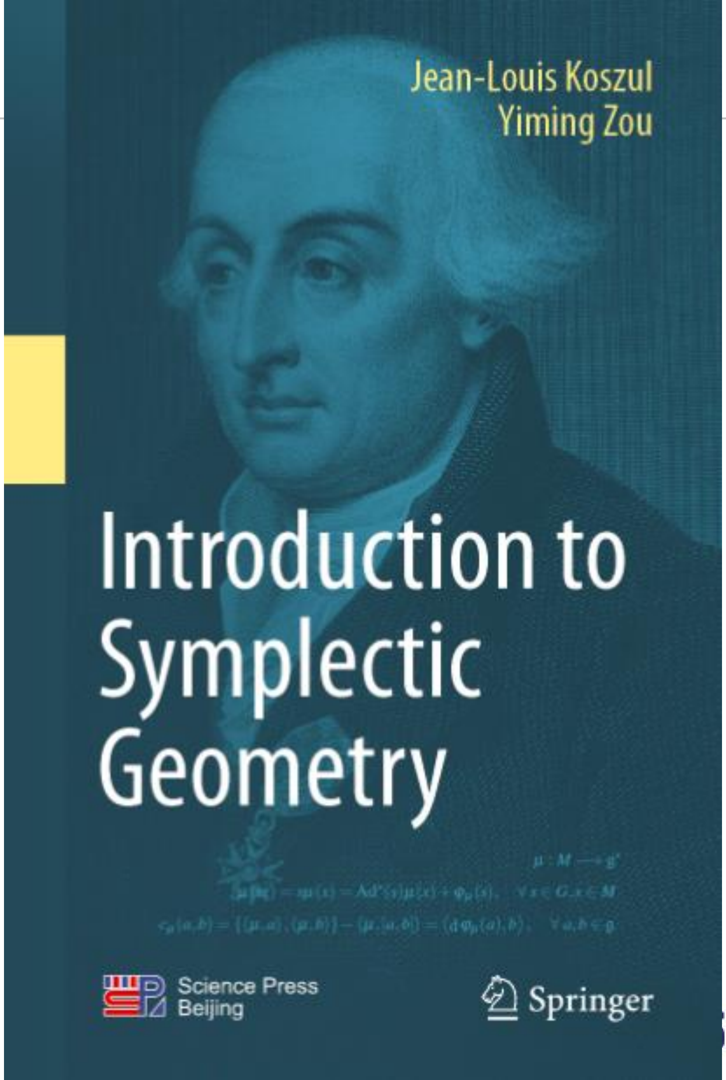
Jean-Louis Koszul · Yiming Zou

Introduction to Symplectic Geometry

Forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle

This introductory book offers a unique and unified overview of symplectic geometry, highlighting the differential properties of symplectic manifolds. It consists of six chapters: Some Algebra Basics, Symplectic Manifolds, Cotangent Bundles, Symplectic G-spaces, Poisson Manifolds, and A Graded Case, concluding with a discussion of the differential properties of graded symplectic manifolds of dimensions (0,n). It is a useful reference resource for students and researchers interested in geometry, group theory, analysis and differential equations.





Massieu Potential versus Gibbs Potentials

GIBBS Potential: Free Energy

$$\cancel{F = E - TS}$$

Joseph Louis François Bertrand gave to François Massieu a bad advice:

(1) Dans le mémoire dont un extrait est inséré aux *Comptes rendus de l'Académie des sciences* du 18 octobre 1869, ainsi que dans la Note additionnelle insérée le 22 novembre suivant, j'avais adopté pour fonction caractéristique $\frac{H}{T}$, ou $S - \frac{U}{T}$; c'est d'après les bons conseils de M. Bertrand que j'y ai substitué la fonction H, dont l'emploi réalise quelques simplifications dans les formules.

MASSIEU Potential : characteristic function

$$\frac{F}{T} = \frac{1}{T}E - S \underset{\beta=\frac{1}{T}}{\Rightarrow} \Phi = \langle \beta, E \rangle - S$$

Preservation of Legendre Duality

[X] Roger Balian, François Massieu and the thermodynamic potentials, Comptes Rendus Physique Volume 18, Issues 9–10, November–December 2017, Pages 526-530 https://www.sciencedirect.com/science/article/pii/S1631070517300671

Preamble: Souriau Lie Groups Thermodynamics

- Lie groups are in common use in robotics, but still seem to be little used in machine learning.

- We present a model from **Geometric Mechanics**, developed by Jean-Marie Souriau as part of **Mechanical Statistics**, allowing to define an **invariant Fisher-type metric** and **covariant statistical densities** under Lie group action.

- This new approach makes it possible to extend supervised and un-supervised machine learning, jointly:
  - **to elements belonging to a (matrix) Lie group**
  - **to elements belonging to a homogeneous manifold** on which a group acts transitively.

- Other models are under study also using the theory of representations of Lie groups [see Tojo keynote].

Preamble: Souriau Lie Group Statistics & Machine Learning

It could be applied for Lie Groups Statistical Analysis for:

- **Rigid Objects Trajectories** via the SE(3) Lie group
- **Articulated objects** via the SO(3) Lie group
- **Moving parts Dynamics** via the SU(1,1) Lie group

The Souriau model makes it possible in particular to define:

- **a Gibbs density of Maximum Entropy** on the Lie group coadjoint orbits (in the dual space of their Lie algebra)
- with **coadjoint orbits** considered as a homogeneous symplectic manifold.

These densities are parameterized via the Souriau "Moment Map" :

- map from the symplectic manifold to the dual space of Lie algebra
- tool geometrizing Noether's theorem
- on which the group acts via the coadjoint operator

Preamble: New Geometric Entropy Definition

- This model is also very useful in control and navigation, because it makes it possible to **extend concept of "Gaussian" noises (in the sense of the maximum Entropy) on the Lie algebra**.

- In this new model, Entropy is defined as **an invariant Casimir function in coadjoint representation** (this fact gives a natural geometrical definition to Entropy via the structural coefficients).

- This Souriau Model of Lie Groups Thermodynamics:
  - Is developed in a MDPI "Entropy" Special Issue on "**Lie Group Machine Learning and Lie Group Structure Preserving Integrators**"
  - will be presented at Les Houches SPIGL'20 on "**Joint Structures and Commun Foundation of Statistical Physics, Information Geometry and Inference for Learning**"
  - will be presented at IRT SystemX workshop on "**Topological and geometric approaches for statistical learning**"

THALES

Motivations for Lie Group Machine Learning

AI/Machine Learning Evolution: ALGEBRA COMPUTATION STRUCTURES

Calcul formel pour les méthodes de Lie en mécanique hamiltonienne
P.V. Koseleff, X/CMLS PhD, 1993 (P. Cartier)

Souriau Exponential Map Algorithm for Machine Learning on Matrix Lie Groups
Frédéric Barbaresco, Springer GSI'19, 2019

Supervarieties, Sow. Math. Dokl. 16 (1975), 1218-1222.
F. A. Berzin and D. A. Leites

LIE SUPER ALGEBRA

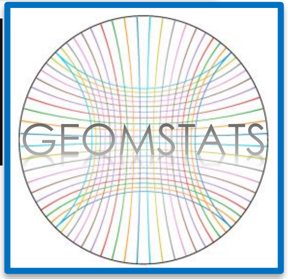
$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

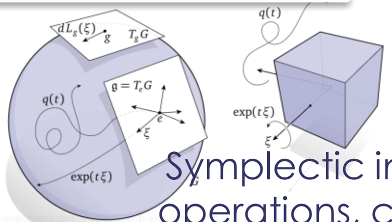
$$\mathrm{Ber}(X) = \det(A)\det(D - CA^{-1}B)^{-1}$$

Berezian Determinant

axiom

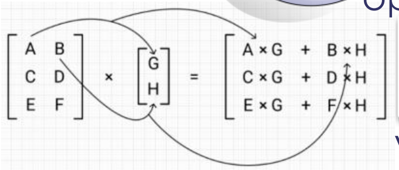
Computer Algebra Group- Scratchpad, IBM,1971





LIE ALGEBRA

Symplectic integrators, non-commutative operations, coadjoint orbits, moment map



LINEAR ALGEBRA

Vectors space, commutative matrix operations, eigen-analysis

GOOGLE TPU (Tensor Processing Unit)

x	y	x AND y	x OR y	NOT x	NOT y
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	1
1	1	1	1	0	0

BOOLE ALGEBRA

Boolean logic digital circuits using electromechanical relays as the switching element.

George R. Stibitz (Bell Labs)

ALGEBRA is the study of mathematical symbols and the rules for manipulating these symbols

THALES

GEOMSTATS: PYTHON Library for Lie Group Machine Learning

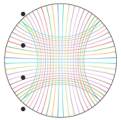
Geomstats

https://github.com/geomstats/geomstats

pypi package 2.1.0 | build passing | codecov 92% | codecov unknown | codecov unknown (Coverages for: numpy, tensorflow, pytorch)

Geomstats is an open-source Python package for computations and statistics on manifolds. The package is organized into two main modules: `geometry` and `learning`.

The module `geometry` implements concepts in differential geometry, and the module `learning` implements statistics and learning algorithms for data on manifolds.



To get started with `geomstats`, see the examples directory.
For more in-depth applications of `geomstats`, see the applications repository.
The documentation of `geomstats` can be found on the documentation website.
If you find `geomstats` useful, please kindly cite our paper.

Install geomstats via pip3

Video: https://m.youtube.com/watch?v=Ju-Wsd84uG0

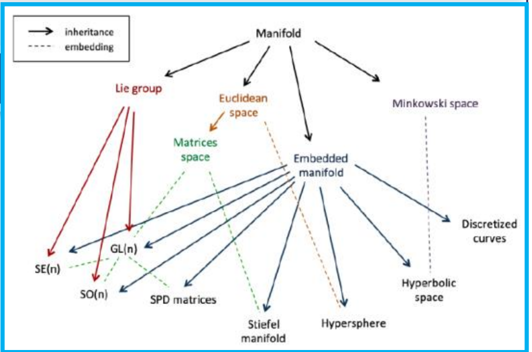
```
pip3 install geomstats
```

hal-02536154, version 1 | Pré-publication, Document de travail

https://hal.inria.fr/hal-02536154

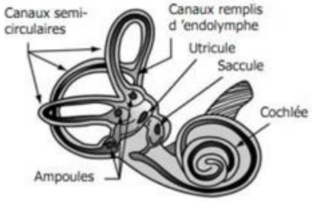
Geomstats: A Python Package for Riemannian Geometry in Machine Learning

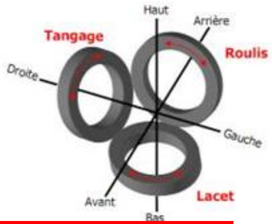
Nina Miolane [1], Alice Le Brigant, Johan Mathe [2], Benjamin Hou [3], Nicolas Guigui [4, 5], Yann Thanwerdas [4, 5], Stefan Heyder [6], Olivier Peltre, Niklas Koep, Hadi Zaatiti [7], Hatem Hajri [7], Yann Cabanes, Thomas Gerald, Paul Chauchat [8], Christian Shewmake, Bernhard Kainz, Claire Donnat [9], Susan Holmes [1], Xavier Pennec [4, 5] Détails



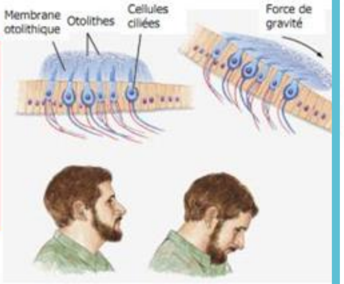
Motivation for Lie Group Machine Learning : Data as Lie Groups

Geolocalization and Navigation : Visio-Inertial SLAM: Visio-Vestibular Brain System

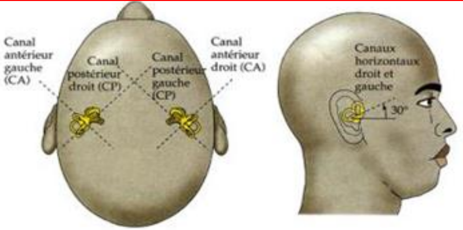


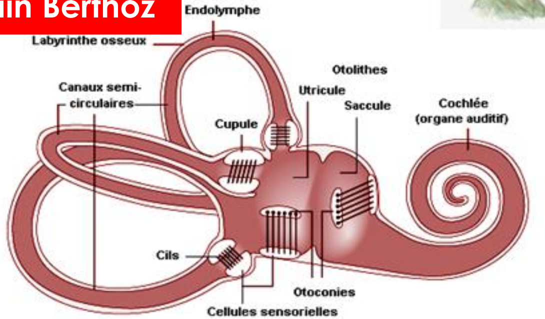


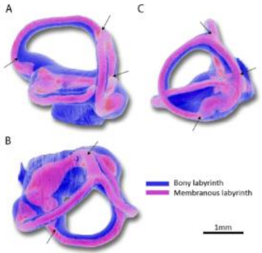
Coding of Homogeneous Galileo Group By Vestibular System and Otolithes

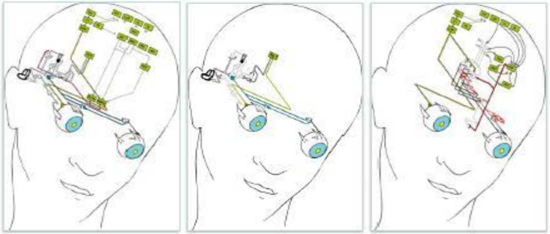


Works of Daniel Bennequin & Alain Berthoz









VINet: Visual-Inertial Odometry as a Sequence-to-Sequence Learning Problem

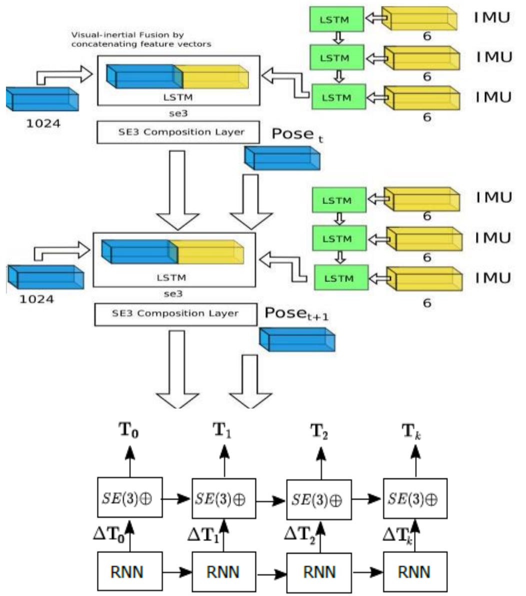
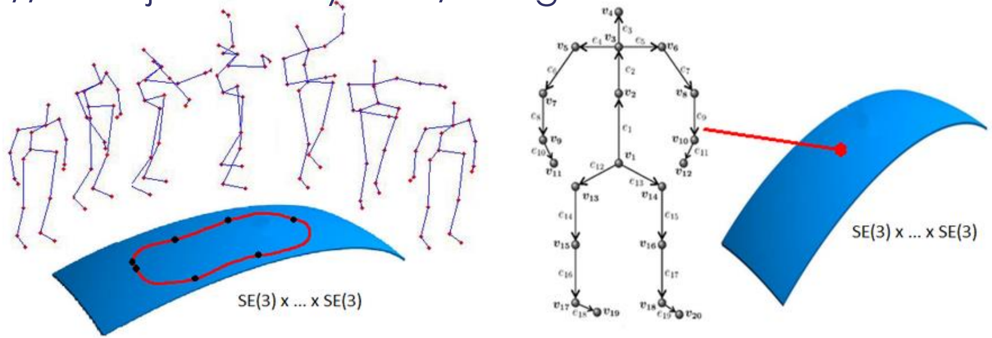


Illustration of the SE(3) composition layer - a parameter-free layer which concatenates transformations between frames on SE(3).

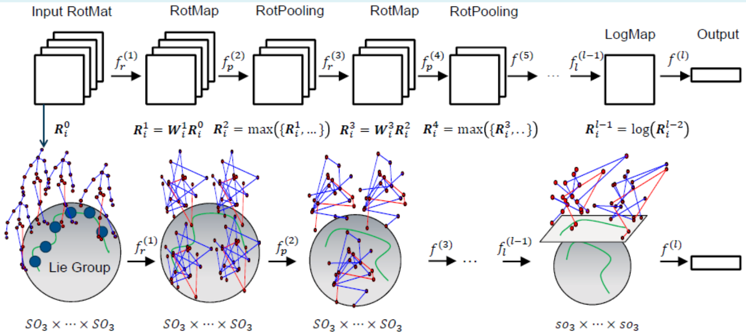
Motivation for Lie Group Machine Learning: Data as Lie Groups

Articulated 3D Movment/Posture Learning

http://ravitejav.weebly.com/rolling.html



Zhiwu Huang, Chengde Wan, Thomas Probst, Luc Van Gool, Deep Learning on Lie Groups for Skeleton-based Action Recognition, Computer Vision and Pattern Recognition, CVPR 2017



$$SO(3)=\left\{\Omega \,/\, \Omega^T\Omega=\Omega\Omega^T=I,\det^2\Omega=1\right\}$$

$$\begin{bmatrix} Z' \\ 1 \end{bmatrix}=\begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z \\ 1 \end{bmatrix},\quad \begin{cases} \Omega\in SO(3) \\ t\in R^3 \end{cases}$$

$$\begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix}\in SE(3)$$

$$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_p \end{bmatrix}\in SO(3)\times\ldots\times SO(3)$$

Vectors of $SE(3)$:

$$\begin{bmatrix} \begin{bmatrix} \Omega_1 & t_1 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} \Omega_2 & t_2 \\ 0 & 1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Omega_m & t_m \\ 0 & 1 \end{bmatrix} \end{bmatrix}\in SE(3)\times\ldots\times SE(3)$$

Path Signatures on Lie Groups

Path Signatures on Lie Groups

Darrick Lee LDARRICK@SAS.UPENN.ED
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104, USA

Robert Ghrist GHRIST@SEAS.UPENN.ED
Departments of Mathematics and Electrical & Systems Engineering
University of Pennsylvania
Philadelphia, PA 19104, USA

Editor:

Abstract

Path signatures are powerful nonparametric tools for time series analysis, shown to form a universal and characteristic feature map for Euclidean valued time series data. We lift the theory of path signatures to the setting of Lie group valued time series, adapting these tools for time series with underlying geometric constraints. We prove that this generalized path signature is universal and characteristic. To demonstrate universality, we analyze the human action recognition problem in computer vision, using $SO(3)$ representations for the time series, providing comparable performance to other shallow learning approaches, while offering an easily interpretable feature set. We also provide a two-sample hypothesis test for Lie group-valued random walks to illustrate its characteristic property. Finally we provide algorithms and a Julia implementation of these methods.

Keywords: path signature, Lie groups, universal and characteristic kernels

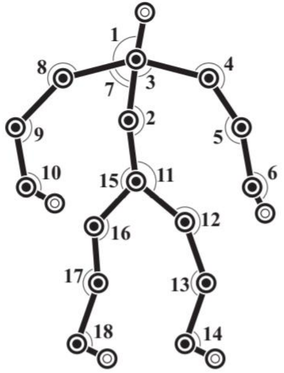
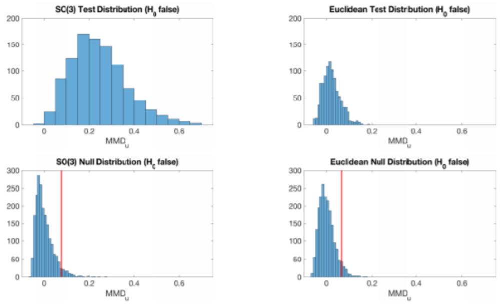


Figure 5: Numbering of the primary pairs of body parts



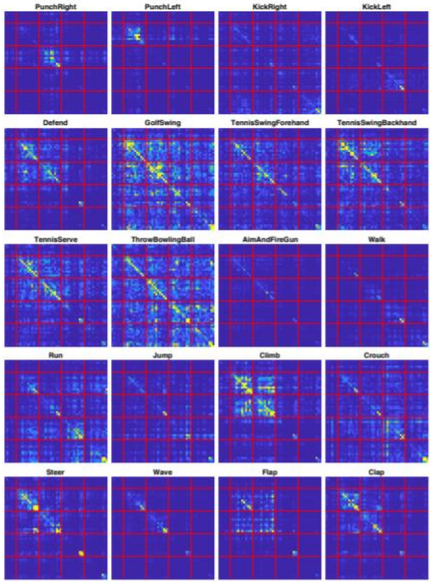


Figure 8: Averaged absolute S^2 matrices for all actions in G3D dataset.

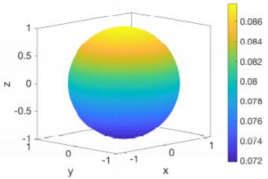
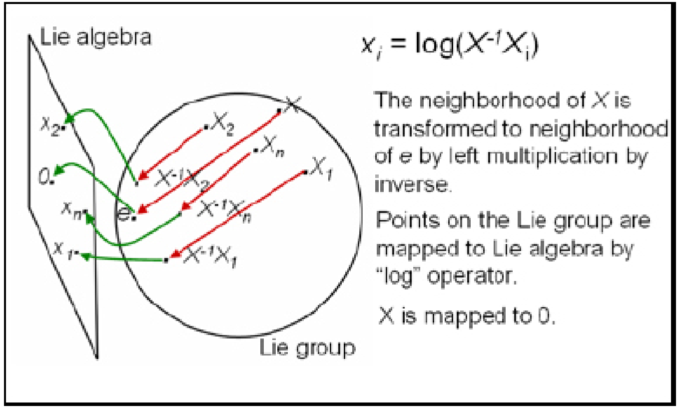


Figure 9: The von-Mises Fisher density on S^2 with mean direction $x = (0, 0, 1)$ and $\kappa = 0.1$.

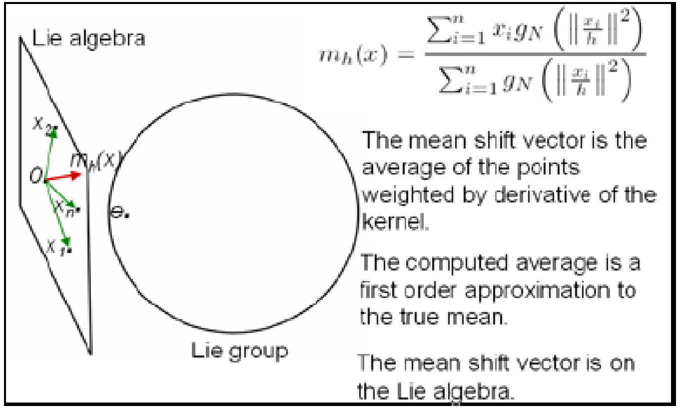
OPEN

THALES

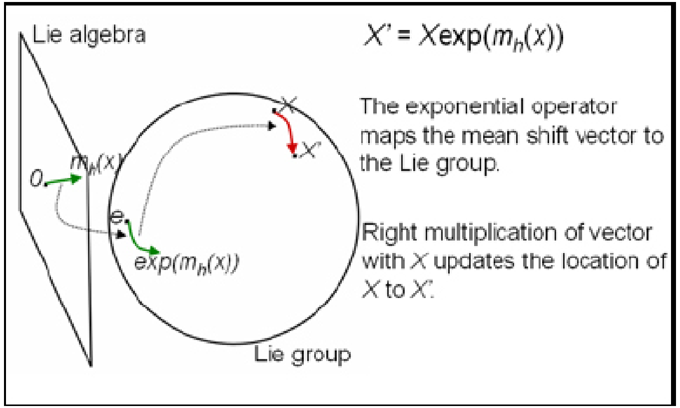
Extension of Mean-Shift for Lie Group (e.g. with SO(3))



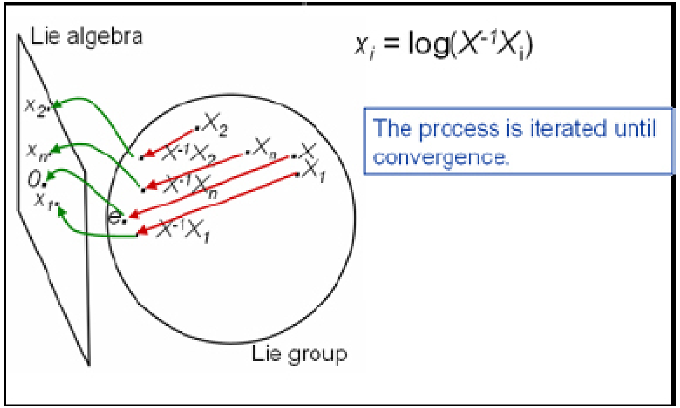
(1)



(2)



(3)



(4)

$$d(X, Y) = \|\log(X^{-1}Y)\|$$

$$k_N(s) = e^{-\frac{1}{2}s}$$

$$\hat{f}(X) = \frac{c_{k,d}}{nh^d}\sum_{i=1}^{n} k_N\left(\left\|\frac{\log(X^{-1}X_i)}{h}\right\|^2\right)$$

$$x_i = \log(X^{-1}X_i)$$

$$m_h(x) = \frac{\sum_{i=1}^{n} x_i g_N\left(\left\|\frac{x_i}{h}\right\|^2\right)}{\sum_{i=1}^{n} g_N\left(\left\|\frac{x_i}{h}\right\|^2\right)}$$

Algorithm: MEAN SHIFT ON LIE GROUPS

Given: Data points on Lie group $\{X_j\}_{j=1..n}$

for $j \leftarrow 1..n$

$X \leftarrow X_j$

repeat

for all data points

$x_i \leftarrow \log(X^{-1}X_i)$

$m_h(x) \leftarrow \frac{\sum_{i=1}^{n} x_i g_N\left(\left\|\frac{x_i}{h}\right\|^2\right)}{\sum_{i=1}^{n} g_N\left(\left\|\frac{x_i}{h}\right\|^2\right)}$

$X \leftarrow X\exp(m_h(x))$

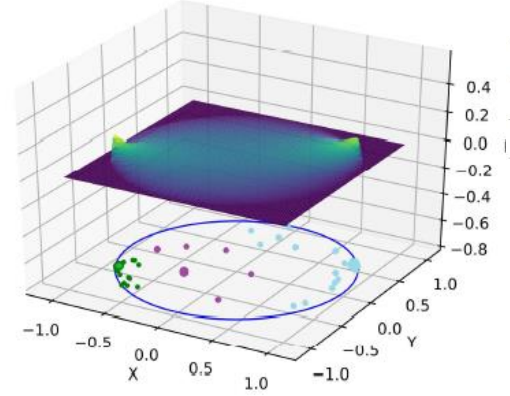
until $m_h(x) < \varepsilon$

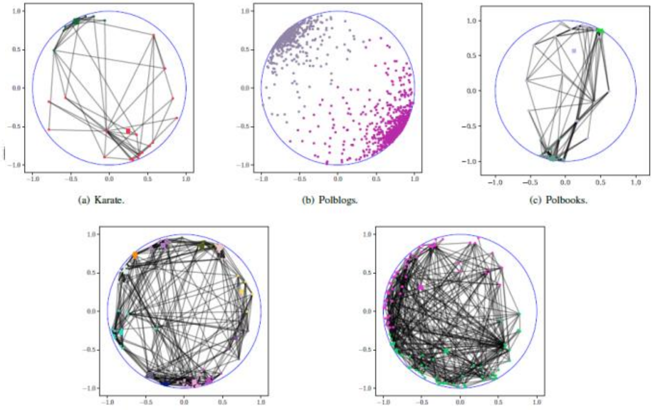
Store X as a local mode.

Report distinct modes.

Motivation for Lie Group Machine Learning: Data in Homogenous Space where a Lie Group acts homogeneously

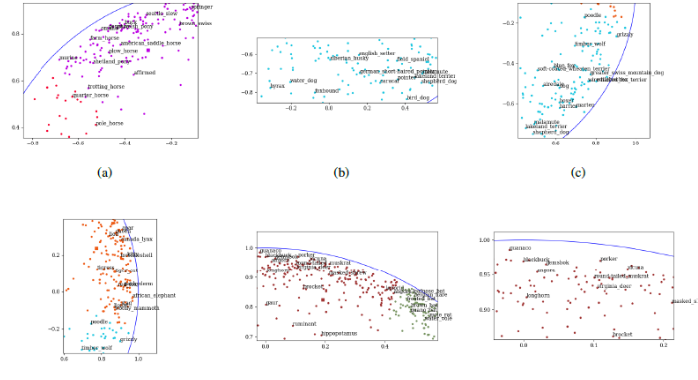
Poincaré/Hyperbolic Embedding in Poincaré Unit Disk for NLP (Natural Langage Processing)





(a) Karate. (b) Polblogs. (c) Polbooks.

(d) Football. (e) Adjnoun.



(a) (b) (c)

(d) (e) (f)

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} / |\alpha|^2 - |\beta|^2 = 1,\ \alpha, \beta \in C \right\}$$

$$D = \{ z = x + iy \in C / |z| < 1 \}$$

$$g(z) = (\alpha z + \beta) / (\beta^* z + \alpha^*)$$

H. Hajri, H. Zaatiti, and G. Hébrail. Learning graph-structured data using poincaré embeddings and riemannian k-means algorithms. CoRR, abs/1907.01662, 2019

M. Nickel and D. Kiela. Poincaré embeddings for learning hierarchical representations. In Advances in Neural Information Processing Systems 30, pages 6338–6347. Curran Associates, Inc., 2017

Image Processing by SE(2) Group: hypoelliptic diffusion in SE(2): analogy with Brain Orientation Maps V1

Image Processing with oriented gradient by SE(2)

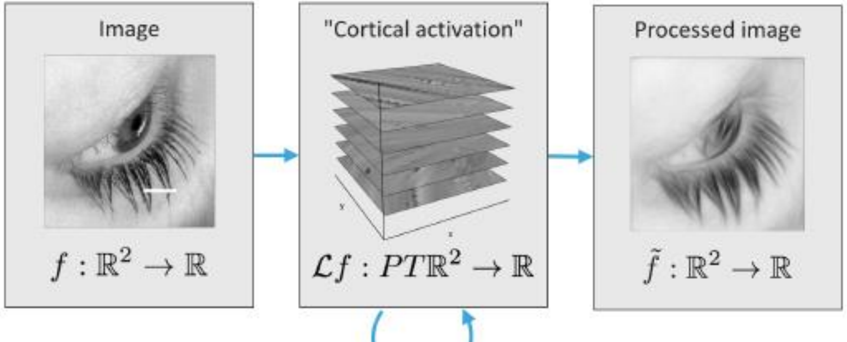
$$\begin{bmatrix} Z' \\ 1 \end{bmatrix} = \begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ 1 \end{bmatrix} \ , \ \begin{cases} \Omega \in SO(2) \\ t \in R^2 \end{cases}$$

$$Z : \mathbb{R}^2 \to \mathbb{C},\ a = \rho e^{i\theta} \mapsto r(a)\, e^{i\varphi(a)}$$

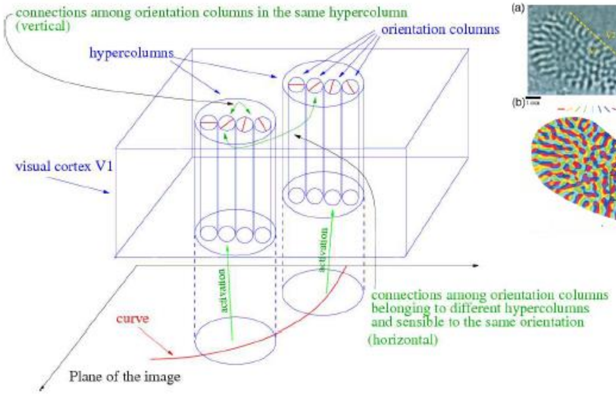
$$\omega_S = -\sin(\theta)\, dx + \cos(\theta)\, dy$$

$$\cos(\theta)(dy - pdx) = \cos(\theta)\,\omega.$$

$$\mathbf{z}_{\exp}(x_0) = \sum_{k=1}^{N} a_{\vartheta_k}(x_0) e^{2i\vartheta_k}$$

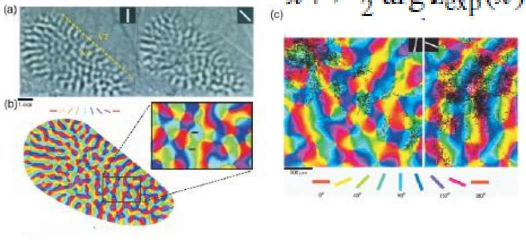


$SE(2)$ double covering of PTR^2



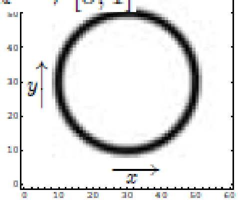
A scheme of the primary visual cortex V1

$$x \mapsto \frac{1}{2} \arg \mathbf{z}_{\exp}(x)$$

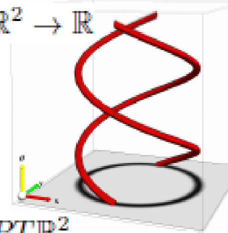


Pinwheel structure of V1

$f : \mathbb{R}^2 \to [0,1]$



$\mathcal{L}f : PT\mathbb{R}^2 \to \mathbb{R}$



$(x, y, \theta) \in PT\mathbb{R}^2$

Lie Group Machine Learning for Drone Recognition

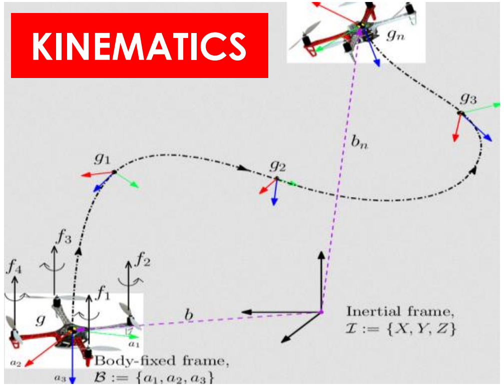
Drone Recognition on Micro-Doppler by SU(1,1) Lie Group Machine Learning

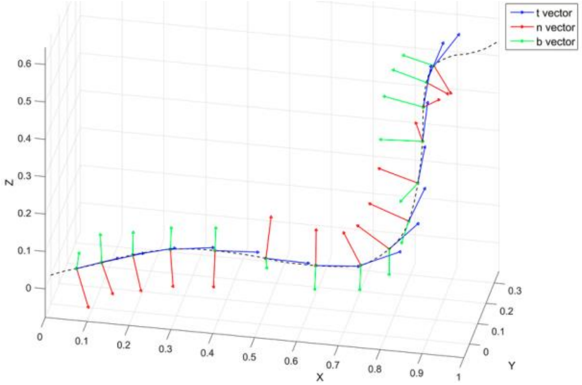
- **Verblunsky/Trench Theorem**: all **Toeplitz Hermitian Positive Definite Covariance matrices** of stationary Radar Time series could be coded and parameterized in a product space with a real positive axis (for signal power) and a **Poincaré polydisk** (for Doppler Spectrum shape).
- **Poincaré Unit Disk is an homogeneous space** where SU(1,1) Lie Group acts transitively. Each data in Poincaré unit disk of this polydisk could be then coded by SU(1,1) matrix Lie group element.
- Micro-Doppler Analysis can be achieved by SU(1,1) Lie Group Machine Learning.

Drone Recognition on Kinematics by SE(3) Lie Group Machine Learning

- **Trajectories** could be coded by SE(3) Lie group time series provided through Invariant Extended Kalman Filter (IEKF) Radar Tracker based on local Frenet-Seret model.
- **Drone kinematics** will be then coded by time series of SE(3) matrix Lie Groups characterizing **local rotation/translation of Frenet frame** along the drone trajectory.

Drone Recognition by Lie Group Machine Learning: SU(1,1) & SE(3)

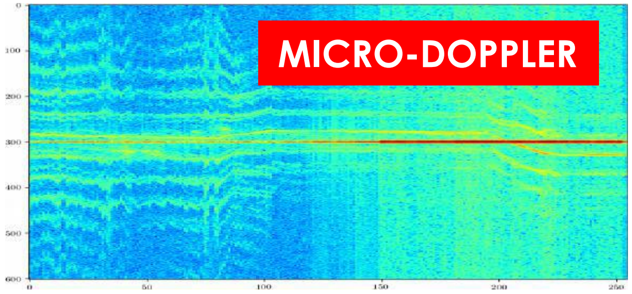


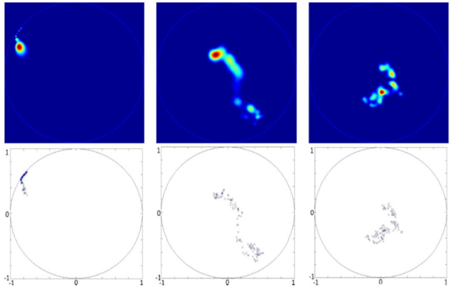


$$SE(3)=\left\{\begin{bmatrix}\Omega & t\\ 0 & 1\end{bmatrix} / \Omega\in SO(3), t\in R^3\right\}$$

$$SO(3)=\left\{\Omega / \Omega^T\Omega=\Omega\Omega^T=I, \det^2\Omega=1\right\}$$

$$\left\{\begin{bmatrix}\Omega_1 & t_1\\ 0 & 1\end{bmatrix},\begin{bmatrix}\Omega_2 & t_2\\ 0 & 1\end{bmatrix},\dots,\begin{bmatrix}\Omega_n & t_n\\ 0 & 1\end{bmatrix}\right\}$$





$$SU(1,1)=\left\{\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix} / |\alpha|^2-|\beta|^2=1, \ \alpha,\beta\in C\right\}$$

$$\left\{\begin{bmatrix}\alpha_1 & \beta_1\\ \beta_1^* & \alpha_1^*\end{bmatrix},\begin{bmatrix}\alpha_2 & \beta_2\\ \beta_2^* & \alpha_2^*\end{bmatrix},\dots,\begin{bmatrix}\alpha_n & \beta_n\\ \beta_n^* & \alpha_n^*\end{bmatrix}\right\}$$

$$\varphi : THDP(n) \to R_+^* \times D^{n-1}$$

$$R_n \mapsto \left(P_0, \mu_1, \dots, \mu_{n-1}\right)$$

$$D=\left\{z=x+iy\in C / |z|<1\right\}$$

OPEN

Motivation for Lie Group Machine Learning: Data in Homogenous Space where a Lie Groups act homogeneously

(Micro-)Doppler & Space-Time wave Learning in Poincaré/Siegel Polydisks

$$SU(1,1)=\left\{\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix} / \alpha,\beta\in C, |\alpha|^2-|\beta|^2=1\right\}$$

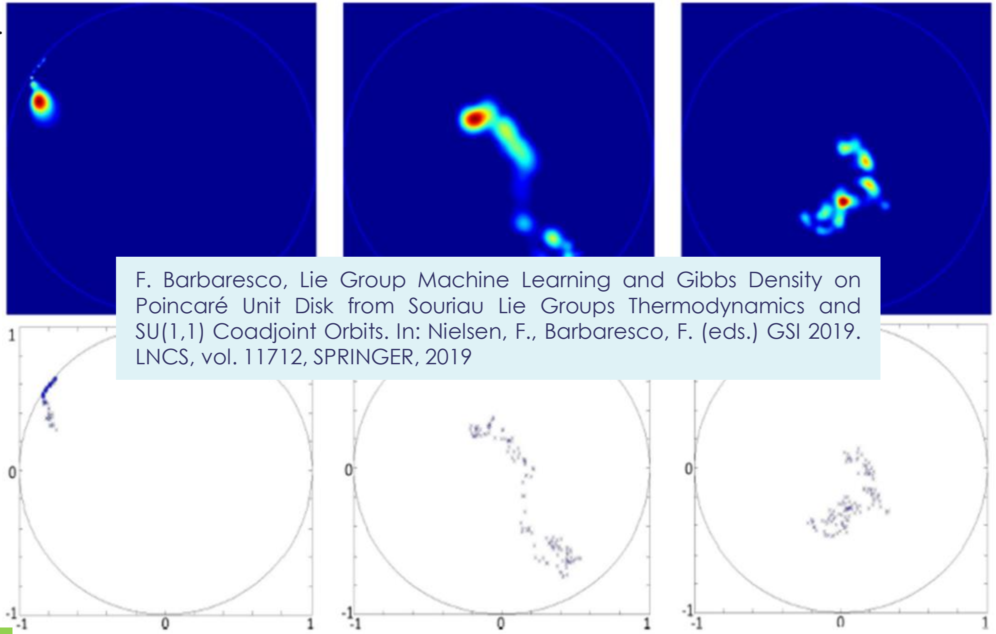
$$\varphi : THDP(n)\to R_+^*\times D^{n-1}$$

$$R_n \mapsto \left(P_0,\mu_1,\ldots,\mu_{n-1}\right)$$

$$R_n=\left(h,\mu_1,\ldots,\mu_{n-1}\right)\in R^{+*}\times D^{n-1}$$

with $\left(D^{n-1}=D\times\ldots\times D\right)$

$$ds^2_{Poincaré}=\frac{|dz|^2}{\left(1-|z|^2\right)^2}=\left(1-zz^*\right)dz\left(1-z^*z\right)dz^*$$



F. Barbaresco, Lie Group Machine Learning and Gibbs Density on Poincaré Unit Disk from Souriau Lie Groups Thermodynamics and SU(1,1) Coadjoint Orbits. In: Nielsen, F., Barbaresco, F. (eds.) GSI 2019. LNCS, vol. 11712, SPRINGER, 2019

$$\varphi : TBTHPD_{n\times n}\to THPD_n\times SD^{n-1}$$

$$R\mapsto\left(R_0,A_1^1,\ldots,A_{n-1}^{n-1}\right)$$

with $SD=\left\{Z\in Herm(n) / ZZ^+<I_n\right\}$

$$ds^2_{Siegel}=Tr\left[\left(I-ZZ^+\right)^{-1}dZ\left(I-Z^+Z\right)^{-1}dZ^+\right]$$

THALES

Matrix Lie Group SU(1,1) for Doppler Data

Lie Group Structure for Doppler Data

- Lie Group structure appears naturally on Doppler data, if we consider time series of locally stationary signal and their associated covariance matrix. Covariance matrix is Toeplitz Hermitian Positive Definite. We can then use a Theorem due to Verblunsky and Trench, that this structure of covariance matrix could be coded in product space involving the Poincaré unit Polydisk:

$$\varphi : THDP(n) \to R_+^* \times D^{n-1}$$
$$R_n \mapsto \left(P_0, \mu_1, \ldots, \mu_{n-1}\right)$$

- where D is the Poincaré Unit Disk: $D = \left\{ z = x + iy \in C \,/\, |z| < 1 \right\}$

Matrix Lie Group SU(1,1) for Doppler Data

Poincaré Unit Disk as a Homogeneous Manifold

- The Poincaré unit disk is an homogeneous bounded domain where the Lie Group SU(1,1) act transitively. This Matrix Group is given by

$$SU(1,1)=\left\{\begin{bmatrix} a & b \\ b^{*} & a^{*}\end{bmatrix} / |a|^2-|b|^2=1,\ a,b\in C\right\}$$

- where SU(1,1) acts on the Poincaré Unit Disk by: $g\in SU(1,1)\Rightarrow g.z=\dfrac{az+b}{b^{*}z+a}$
- with Cartan Decomposition of SU(1,1):

$$\begin{pmatrix} a & b \\ b^{*} & a^{*}\end{pmatrix}=|a|\begin{pmatrix} 1 & z \\ z^{*} & 1\end{pmatrix}\begin{pmatrix} a/|a| & 0 \\ 0 & a^{*}/|a|\end{pmatrix}$$

$$\text{with } z=b\left(a^{*}\right)^{-1}, |a|=\left(1-|z|^2\right)^{-1/2}$$

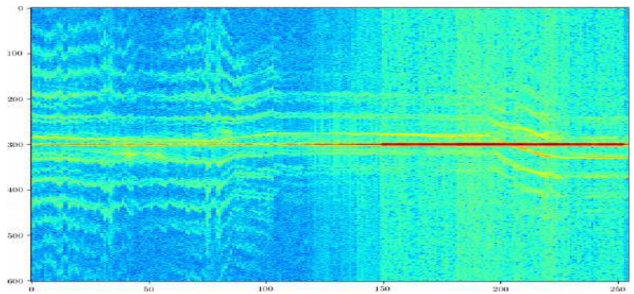
- We can observe that $z=b\left(a^{*}\right)^{-1}$ could be considered as action of $g\in SU(1,1)$ on the centre on the unit disk $z=g.0=b\left(a^{*}\right)^{-1}$.

Les Houches 27th-31st July 2020
Joint Structures and Common Foundations of Statistical Physics,
Information Geometry and Inference for Learning (SPIGL'20)

Matrix Lie Group SU(1,1) for Doppler Data

Coding Doppler Spectrum by data on SU(1,1) Lie group

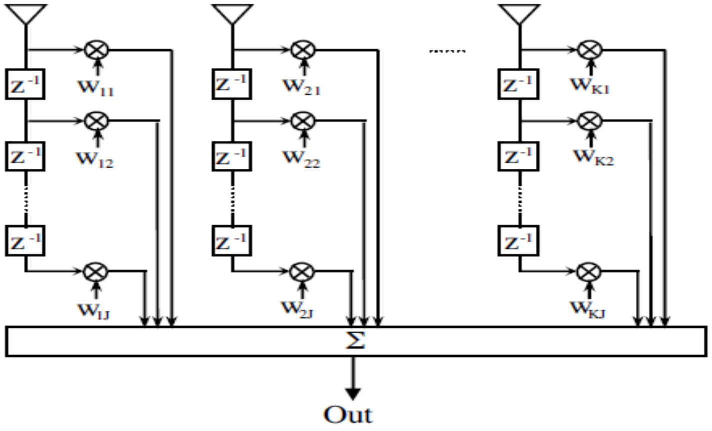
- The principal idea is that we can code any point $z = b\left(a^*\right)^{-1}$ in the unit disk by an element of the Lie Group SU(1,1). Main advantage is that the point position is no longer coded by coordinates but intrinsically by transformation from 0 to this point. Finally, a covariance matrix of a stationary signal could be coded by (n-1) Matrix SU(1,1) Lie Group elements:



$$\mathrm{THPD} \rightarrow R_+^* \times D^{n-1} \qquad \rightarrow R_+^* \times SU(1,1)^{n-1}$$

$$R_n \mapsto \left(P_0, \mu_1, \ldots, \mu_{n-1}\right) \mapsto \left(P_0, \begin{bmatrix} a_1 & b_1 \\ b_1^* & a_1^* \end{bmatrix}, \ldots, \begin{bmatrix} a_{n-1} & b_{n-1} \\ b_{n-1}^* & a_{n-1}^* \end{bmatrix}\right)$$

Extension for Space-Time Processing: Siegel Disk



$$Z = \begin{bmatrix} z_{1,1} \\ \vdots \\ z_{N,1} \\ \vdots \\ z_{1,M} \\ \vdots \\ z_{N,M} \end{bmatrix} \Rightarrow R = E\left[ZZ^{+}\right] = \begin{bmatrix} R_0 & R_1 & \cdots & R_n \\ R_1^{+} & R_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_1 \\ R_n^{+} & \cdots & R_1^{+} & R_0 \end{bmatrix}$$

Matrix Extension of Trench/Verblunsky Theorem: Existence of diffeomorphism φ and Siegel Polydisk (matrix extension of Poincaré Disk)

$$\varphi : TBTHPD_{n\times n} \to THPD_n \times SD^{n-1}$$

$$R \mapsto \left(R_0, A_1^1, \ldots, A_{n-1}^{n-1}\right)$$

$$\text{with} \quad SD = \left\{Z \in Herm(n) / ZZ^{+} < I_n\right\}$$

$$ds^2_{Siegel} = Tr\left[\left(I - ZZ^{+}\right)^{-1} dZ \left(I - Z^{+}Z\right)^{-1} dZ^{+}\right]$$

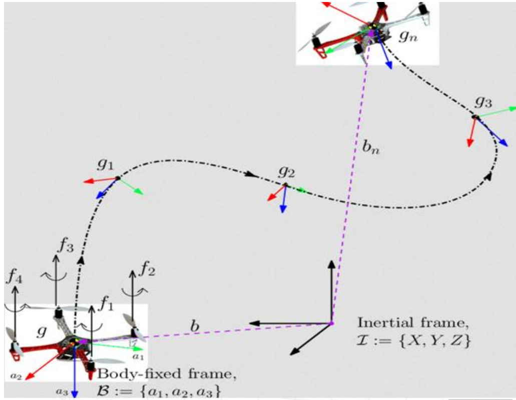
$$ds^2_{Poincaré} = \frac{|dz|^2}{\left(1-|z|^2\right)^2} = \left(1 - zz^*\right) dz \left(1 - z^*z\right) dz^*$$

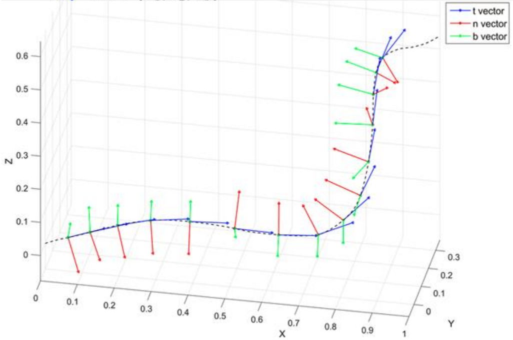
Matrix Lie Group SE(3) for Kinematic Data

3D trajectory and Frenet-Serret Frame

- When we consider a 3D trajectory of a mobile target, we can describe this curve by a time evolution of the local Frenet–Serret frame (local frame with tangent vector, normal vector and binormal vector). This frame evolution is described by the Frenet-Serrtet formula that gives the kinematic properties of the target moving along the continuous, differentiable curve in 3D Euclidean space $\mathbb{R}^3$. More specifically, the formulas describe the derivatives of the so-called tangent, normal, and binormal unit vectors in terms of each other.

$$\frac{d}{dt}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix} \quad \text{with} \quad \begin{cases}\kappa : \text{curvature} \\ \gamma : \text{torsion}\end{cases}$$





Matrix Lie Group SE(3) for Kinematic Data

3D trajectory curve

- we will consider motions determined by exponentials of paths in the Lie algebra. Such a motion is determined by a unit speed space-curve $\tau(t)$. Now in a Frenet-Serret motion a point in the moving body moves along the curve and the coordinate frame in the moving body remains aligned with the tangent $\vec{t}$, normal $\vec{n}$, and binormal $\vec{b}$, of the curve. Using the 4-dimensional representation of the Lie Group SE(3), the motion can be specified as :

$$G(t) = \begin{pmatrix} R(t) & \tau(t) \\ 0 & 1 \end{pmatrix} \in SE(3)$$

- where $\tau(t)$ is the curve and the rotation matrix has the unit vectors $\vec{t}$, $\vec{n}$,and $\vec{b}$ as columns:

$$R(t) = \begin{pmatrix} \vec{t} & \vec{n} & \vec{b} \end{pmatrix} \in SO(3)$$

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Matrix Lie Group SE(3) for Kinematic Data

Time evolution of Frenet-Serret Frame

- If we introduce the Darboux vector $\vec{\omega}=\gamma\vec{t}+\kappa\vec{b}$ that we can rewritte from Frenet-Serret Formulas : $$\frac{d\vec{t}}{dt}=\vec{\omega}\times\vec{t}\ ,\ \frac{d\vec{n}}{dt}=\vec{\omega}\times\vec{n}\ ,\ \frac{d\vec{b}}{dt}=\vec{\omega}\times\vec{b}$$
- Then, we can write with Ω is the 3×3 anti-symmetric matrix corresponding to $\vec{\omega}$: $$\frac{dR}{dt}=\Omega R$$
- We note that $\frac{d\tau(t)}{dt}=\vec{t}$ and $\frac{d\vec{\omega}}{dt}=\frac{d\gamma}{dt}\vec{t}+\frac{d\kappa}{dt}\vec{b}$
- The instantaneous twist of the motion $G(t)$ is given by:

$$S_d=\frac{dG(t)}{dt}G^{-1}(t)=\begin{pmatrix}\Omega & \upsilon\\ 0 & 0\end{pmatrix}$$

Matrix Lie Group SE(3) for Kinematic Data

Instantaneous twist

- This is the Lie algebra element corresponding to the tangent vector to the curve $G(t)$. It is well known that elements of the Lie algebra $se(3)$ can be described as lines with a pitch. The fixed axode of a motion $G(t) \in SE(3)$ is given by the axis of S_d as t varies. The instantaneous twist in the moving reference frame is given by $S_b = G^{-1}(t) S_d G(t)$, that is, by the adjoint action on the twist in the fixed frame. The instantaneous twist S_b can also be found from the relation:

$$S_b = G^{-1}(t)\frac{dG(t)}{dt}$$

$$S_b = G^{-1}\frac{dG}{dt} = \begin{pmatrix} R^T & -R^T\tau \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Omega R & \vec{t} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R^T\Omega R & R\vec{t} \\ 0 & 0 \end{pmatrix}$$

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Matrix Lie Group SE(3) for Kinematic Data

Trajectory as a time series of Matrix SE(3) Lie groups

- We can observe that we could describe a 3D trajectory by a time series of SE(3) Lie group elements:

$$SE(3)=\left\{\begin{bmatrix} R & \tau \\ 0 & 1 \end{bmatrix} / R\in SO(3), \tau\in R^3\right\}$$

with $SO(3)=\left\{R / R^T R = RR^T = I, \det^2 R = 1\right\}$

- Then, the trajectory will be given by the following time series :

$$\left\{\begin{bmatrix} R_1 & \tau_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} R_2 & \tau_2 \\ 0 & 1 \end{bmatrix}, \dots, \begin{bmatrix} R_n & \tau_n \\ 0 & 1 \end{bmatrix}\right\} \in SE(3)^n$$

Lie Group Co-adjoint Orbits & Homogeneous Symplectic Manifold

Structuring Principles for Learning : Calculus of Variations

Pierre de Fermat

Fermat's principle of least time

Pierre Louis Maupertuis

Maupertuis's principle of least length

Joseph Louis Lagrange

(Euler) Lagrange Equation

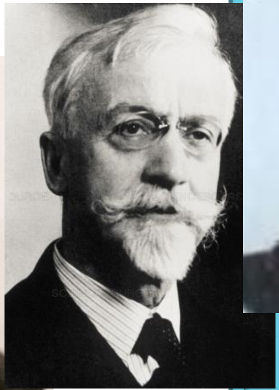
Simeon Denis Poisson

Poisson Bracket, Poisson Geometry Structure

Henri Poincaré

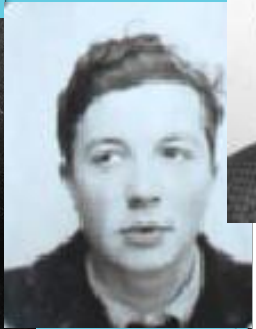
(Euler) Poincaré Equation

Elie Cartan



Poincaré Cartan Integral Invariant

Jean- Marie Souriau



Souriau Moment Map, Souriau Symplectic 2 Form, Lie Groups Thermodynamics

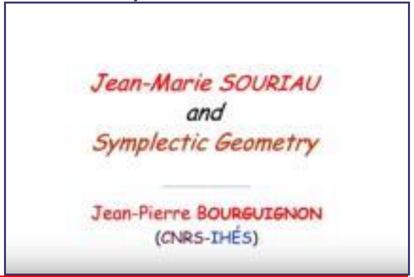
Jean-Michel Bismut

Random Mechanics

SOURIAU 2019

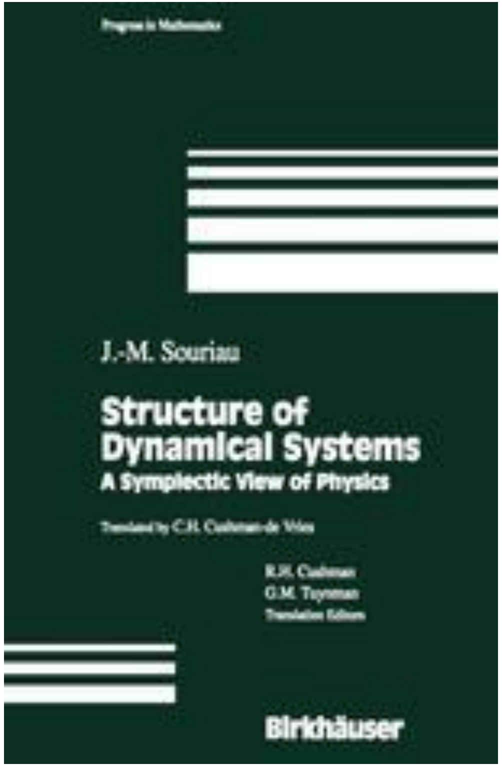
SOURIAU 2019

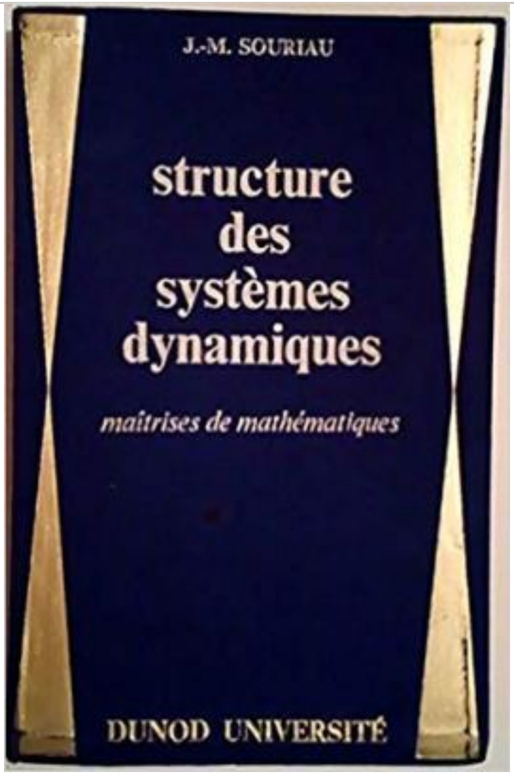
- Internet website : http://souriau2019.fr
- In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**
- Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)
- Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES, director of the European ERC)



https://www.youtube.com/watch?v=93hFolIBo0Q&t=3s

Le Livre de J.M. Souriau « Structure des systèmes dynamiques », 1969





- Introduction of symplectic geometry in mechanics
- Invention of the "moment map"
- Geometrization of Noether's theorem
- Barycentric decomposition theorem
- The total mass of an isolated dynamic system is the class of cohomology of the default of equivariance for the moment map
- Lie Groups Thermodynamics (Chapter IV)

http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm
http://www.springer.com/us/book/9780817636951

Lagrange 2-form rediscovered by Jean-Marie Souriau

- Rewriting equations of classical mechanics in phase space $\begin{pmatrix} r \\ v \end{pmatrix}$

$$m\frac{d^2r}{dt^2}=F \quad \Longrightarrow \quad m\frac{dv}{dt}=F \quad \text{et} \quad v=\frac{dr}{dt}$$

- Souriau rediscovered that Lagrange had considered the evolution space: $y=\begin{pmatrix} t \\ r \\ v \end{pmatrix}\in V$

$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

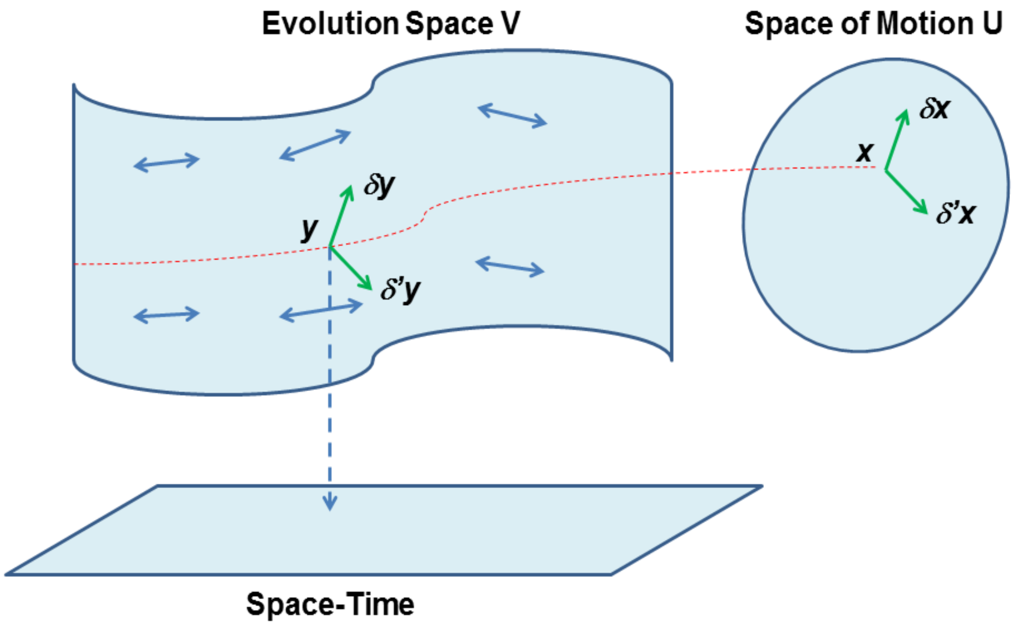
- A dynamic system is represented by a foliation. This foliation is determined by an antisymmetric covariant 2nd order tensor σ , called the Lagrange (-Souriau) form, a bilinear operator on the tangent vectors of V.

$$\sigma(\delta y)(\delta' y)=\langle m\delta v - F\delta t, \delta' r - v\delta' t\rangle - \langle m\delta' v - F\delta' t, \delta r - v\delta t\rangle \qquad \delta y=\begin{pmatrix} \delta t \\ \delta r \\ \delta v \end{pmatrix} \quad \text{et} \quad \delta' y=\begin{pmatrix} \delta' t \\ \delta' r \\ \delta' v \end{pmatrix}$$

- *In the Lagrange-Souriau model, σ is a 2-form on the evolution space V, and the differential equation of motion implies: $\delta y \in \varepsilon$*

$$\sigma(\delta y)(\delta' y)=0 \ , \ \forall \delta' y \qquad\qquad \sigma(\delta y)=0 \ \text{ ou } \ \delta y \in \ker(\sigma)$$

Evolution space of Lagrange-Souriau



$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

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Gallileo Group & Algebra & V. Bargman Central extensions

- Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.

- Gallileo Lie Group & Algebra

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

$\vec{x}, \vec{u}$ and $\vec{w} \in R^3, e \in R^+$

$R \in SO(3)$

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

- Bargmann Central extension:

$$\begin{bmatrix} R & \vec{u} & 0 & \vec{w} \\ 0 & 1 & 0 & e \\ -\vec{u}^t R & -\dfrac{\|\vec{u}\|^2}{2} & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Souriau Work Roots: François Gallissot Theorem

- **Gallissot Theorem**: There are 3 types of differential forms generating the equations of a material point motion, **invariant by the action of the Galileo group**

$$A:\begin{cases} s=\dfrac{1}{2m}\sum_{i=1}^{3}\left(mdv_i - F_i dt\right)^2 \\ e=\dfrac{m}{2}\sum_{j=1}^{3}\left(dx_j - v_j dt\right)^2 \end{cases}$$

F. Gallissot, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

$$B: f=\sum_{1}^{3}\delta_{ij}\left(dx_i - v_i dt\right)\left(mdv_j - F_j dt\right) \text{ with } \delta_{ij} \text{ krönecker symbol}$$

$$C:\ \omega=\sum_{1}^{3}\delta_{ij}\left(mdv_i - F_i dt\right)\wedge\left(dx_j - v_j dt\right)$$

- $d\omega=0$ constrained the Pfaff form $\delta_{ij}F_i dx_j$ to be closed and to be reduced to the differential of U : $C\Rightarrow \omega = m\delta_{ij}dv_i\wedge dx_j - dH\wedge dt$ with $H=T-U$ and $T=1/2\sum_{i=1}^{3}m\left(v_i\right)^2$
- It proves that ω has an exterior differential $d\omega$ generating **Poincaré-Cartan Integral invariant:**

$$d\omega=\sum_{i=1}^{3}mv_i dx_j - Hdt$$

François Gallissot Work in 1952 based on Elie and Henri Cartan works

LES FORMES EXTÉRIEURES EN MÉCANIQUE

par F. GALLISSOT.

1952

INTRODUCTION

La mécanique des systèmes paramétriques développée traditionnellement d'après les idées de Lagrange s'est toujours heurtée à des difficultés notables lorsqu'elle a désiré aborder les questions de frottement entre solides (impossibilité et indétermination) ou la notion générale de liaison (asservissement de M. Béghin), d'autre part la forme lagrangienne des équations du mouvement ne nous donne aucune indication sur la nature du problème de l'intégration.

Dans ces célèbres leçons sur les invariants intégraux Élie Cartan a montré que toutes les propriétés des équations différentielles de la dynamique des sytèmes holonomes résultaient de l'existence de l'invariant intégral $\int \omega$, $\omega = p_i dq^i - Hdt$. Ainsi à tout système holonome dont les forces dérivent d'une fonction de forces est associé une forme ω, les équations du mouvement étant les caractéristiques de la forme extérieure $d\omega$. Au cours de ces dix dernières années, sous l'influence des topologistes s'est édifiée sur des bases qui semblent définitives la théorie des formes extérieures sur les variétés différentiables. Il est alors naturel de se demander si la mécanique classique ne peut pas bénéficier largement de ce courant d'idées, si elle ne peut pas être construite en plaçant à sa base une forme extérieure de degré deux, si grâce à la notion de variétés, la notion de liaison ne peut pas être envisagée sous un angle plus intelligible, si les indéterminations et impossibilités qui paraissent paradoxales dans le cadre lagrangien n'ont pas une explication naturelle, enfin s'il n'est pas possible de considérer sous un jour nouveau le problème de l'intégration des équations du mouvement, ces dernières étant engendrées par une forme Ω de degré deux.

S'affranchir de la servitude des coordonnées

$$i(E)\Omega = 0$$

Pour atteindre ces divers objectifs il m'a semblé utile de reprendre dans le chapitre I l'étude des bases logiques sur lesquelles est édifiée la mécanique galiléenne. Je montre ainsi dans le § 1 que lorsqu'on se propose de trouver des formes génératrices des équations du mouvement d'un point matériel invariantes dans les transformations du groupe galiléen, la forme la plus intéressante est une forme extérieure de degré deux définie sur une variété $V_7 = E_3 \otimes E_3 \otimes T$ (E_3 espace euclidien, T droite numérique temporelle)(¹). Dans le § 11 on montre qu'à tout système paramétrique holonome à n degrés de liberté est associé une forme Ω de degré deux de rang $2n$ définie sur une variété différentiable dont les caractéristiques sont les équations du mouvement (²). Cette forme s'exprime si l'on veut au moyen de $2n$ formes de Pfaff et de dt, la forme hamiltonienne n'étant qu'un cas particulier simple. Dans le § 3 j'indique sommairement comment on peut s'affranchir de la servitude des coordonnées dans l'étude des systèmes dynamiques et le rôle important joué par l'opérateur $i(\;)$ antidérivation de M. H. Cartan (³), le champ caractéristique E de la forme Ω étant défini par la relation $i(E)\Omega = 0$.

(¹) M. Kravtchenko a présenté cette conception au VIIIe Congrès de Mécanique.

(²) Dès 1946 M. Lichnerowicz au *Bulletin des Sciences Mathématiques* tome LXX, p. 90 a déjà introduit les formes extérieures pour la formation des équations des systèmes holonomes et linéairement non holonomes.

(³) M. H. Cartan, Colloque de Topologie, Bruxelles, 1950. Masson, Paris, 1951.

F. Gallissot, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

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Interior/Exterior Products and Lie derivative

- $i_V\omega$ is the (p-1)-form on X obtained by inserting $V(x)$ as the first argument of ω :

Interior product : $i_V\omega\left(v_2,\cdots v_p\right)=\omega\left(V(x),v_2,\cdots,v_p\right)$

- $\theta\wedge\omega$ is the (p + 1)-form on X where ω is a p-form and θ is a 1-form on X:

Exterior product : $\theta\wedge\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\,\theta(v_i)\omega\left(v_0,\cdots,\hat{v}_i,\cdots,v_p\right)$

(where the hat indicates a term to be omitted).

- $L_V\omega$ is a p-form on X ,and $L_V\omega=0$ if the flow of V consists of symmetries of ω:

Lie derivative : $L_V\omega\left(v_1,\cdots,v_p\right)=\left.\dfrac{d}{dt}e^{tV*}\omega\left(v_1,\cdots,v_p\right)\right|_{t=0}$

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Exterior derivative and E.Cartan, H. Cartan & S. Lie formulas

- $d\omega$ is the (p+1)-form on X defined by taking the ordinary derivative of ω and then antisymmetrizing:

Exterior derivative : $d\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}(-1)^i\frac{\partial\omega}{\partial x}(v_i)(v_0,\cdots,\hat{v}_i,\cdots,v_p)$

$$p=0,[d\omega]_i=\partial_i\omega \;\;;\;\; p=1,[d\omega]_{ij}=\partial_i\omega_j-\partial_j\omega_i \;\;;\;\; p=2,[d\omega]_{ijk}=\partial_i\omega_{jk}+\partial_j\omega_{ki}+\partial_k\omega_{ij}$$

- The properties of the exterior and Lie Derivative are the following:

$L_V\omega=di_V\omega+i_Vd\omega$ (E. Cartan)

$i_{[U,V]}\omega=i_VL_U\omega-L_Ui_V\omega$ (H.Cartan)

$L_{[U,V]}\omega=L_VL_U\omega-L_UL_V\omega$ (S. Lie)

Souriau Moment Map (1/2)

- Let (X,σ) be a connected symplectic manifold.
- A vector field η on X is called symplectic if its flow preserves the 2-form :

$$L_\eta \sigma = 0$$

- If we use Elie Cartan's formula, we can deduce that :

$$L_\eta \sigma = di_\eta \sigma + i_\eta d\sigma = 0$$

- but as $d\sigma = 0$ then $di_\eta \sigma = 0$. We observe that the 1-form $i_\eta \sigma$ is closed.
- When this 1-form is exact, there is a smooth function $x \mapsto H$ on X with:

$$\boxed{i_\eta \sigma = -dH}$$

- This vector field η is called Hamiltonian and could be define as s symplectic gradient :

$$\boxed{\eta = \nabla_{Symp} H}$$

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Souriau Moment Map (2/2)

$$d i_\eta \sigma = 0 \qquad \boxed{i_\eta \sigma = -dH}$$

- We define the Poisson bracket of two functions H , H' by :

$$\{H, H'\} = \sigma(\eta, \eta') = \sigma\left(\nabla_{Symp} H', \nabla_{Symp} H\right)$$

with $i_\eta \sigma = -dH$ and $i_{\eta'} \sigma = -dH'$

- Let a Lie group G that acts on X and that also preserve σ .

- A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map exists:

$\boxed{\Phi : X \to \mathfrak{g}^*}$ with $\boxed{i_{Z_X} \sigma = -dH_Z}$ where $\boxed{H_Z = \langle \Phi(x), Z \rangle}$

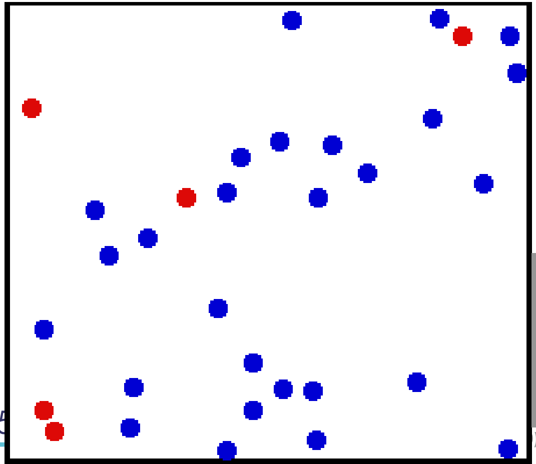
Souriau Model of Lie Groups Thermodynamics

- Souriau Geometric (Planck) Temperature is **an element of Lie Algebra** of Dynamical Group (Galileo/Poincaré groups) acting on the system

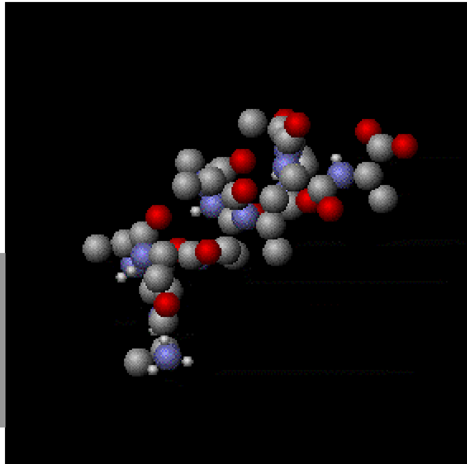
- Generalized Entropy is **Legendre Transform of minus logarithm of Laplace Transform**

- Fisher(-Souriau) Metric is a **Geometric Calorific Capacity (hessian of Massieu Potential)**

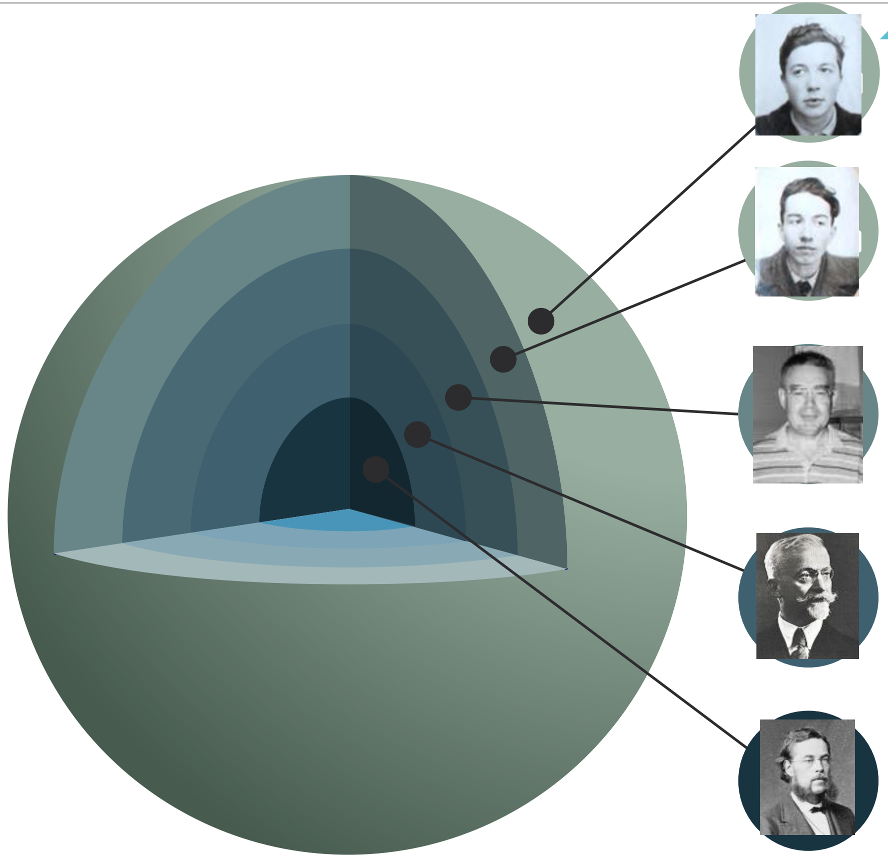
- Higher Order Souriau Lie Groups Thermodynamics is given by **Günther's Poly-Symplectic Model (vector-valued model in non-equivariant case)**



Souriau formalism is fully **covariant**, with no special coordinates (**covariance of Gibbs density wrt Dynamical Groups**)



Lie Groups Tools Development: From Group to Co-adjoint Orbits



Lie Group & Statistical Physics

Jean-Michel Bismut – Random Mechanics
Jean-Marie Souriau – Lie Group Thermodynamics, Souriau Metric
Jean-Louis Koszul – Affine Lie Group & Algebra representation

Harmonic Analysis on Lie Group & Orbits Method

Pierre Torasso & Michèle Vergne – Poisson-Plancherel Formula
Michel Duflo – Extension of Orbits Method, Plancherel & Character
Alexandre Kirillov – Coadjoint Orbits, Kirillov Character
Jacques Dixmier – Unitary representation of nilpotent Group

Lie Group Representation

Bertram Kostant – KKS 2-form, Geometric Quantization
Alexandre Kirillov – Representation Theory, KKS 2-form
Jean-Marie Souriau – Moment Map, KKS 2-form, Souriau Cocycle
Valentine Bargmann – Unitary representation, Central extension

Lie Group Classification

Carl-Ludwig Siegel – Symplectic Group
Hermann Weyl – Conformal Geometry, Symplectic Group
Elie Cartan – Lie algebra classification, Symmetric Spaces
Willem Killing – Cartan-Killing form, Killing Vectors

Group/Lie Group Foundation

Henri Poincaré – Fuchsian Groups
Felix Klein – Erlangen Program (Homogeneous Manifold)
Sophus Lie – Lie Group
Evariste Galois/Louis Joseph Lagange – Substitution Group

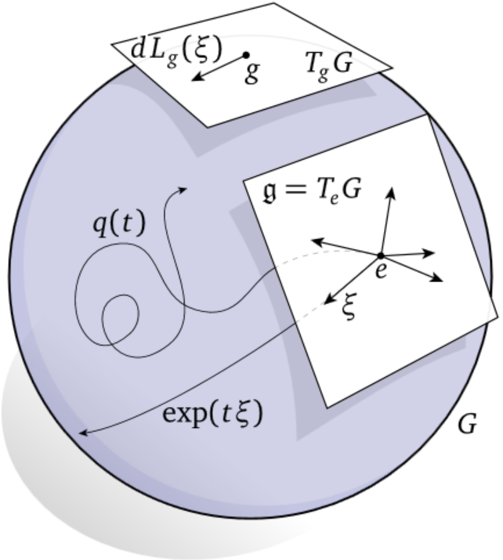
Lie Group

GROUP (Mathematics)

A set equipped with a binary operation with 4 axioms:

- Closure $\forall a,b \in G \text{ then } a \bullet b \in G$
- Associativity $\forall a,b,c \in G \text{ then } (a \bullet b) \bullet c = a \bullet (b \bullet c)$
- Identity $\exists e \in G \text{ such that } e \bullet a = a \bullet e = a$
- invertibility $\forall a \in G, \exists b \in G \text{ such that } b \bullet a = a \bullet b = e$

LIE GROUP



- A group that is a differentiable manifold, with the property that the group operations of multiplication and inversion are smooth maps:

$$\forall x,y \in G \text{ then } \phi : G \times G \to G \text{ then } \phi(x,y) = x^{-1}y \text{ is smooth}$$

- A Lie algebra $\mathfrak{g} = T_e G$ is a vector space with a binary operation called the Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies axioms:

$$[ax+by,z] = a[x,z] + b[y,z] \;\; ; \;\; [x,x] = 0 \;\; ; \;\; [x,y] = -[y,x]$$

$$\text{Jacobi Identity: } [x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

$$[x,y] = xy - yx \text{ for Matrix Lie Group}$$

Lie Group Notation

Lie Algebra of Lie Group and Adjoint operators

- Let G a Lie Group and T_eG tangent space of G at its neutral element e

  - Ad Adjoint representation of G

$$Ad : G \to GL(T_eG) \quad \text{with} \quad i_g : h \mapsto ghg^{-1}$$

$$g \in G \mapsto Ad_g = T_e i_g$$

  - ad Tangent application of Ad at neutral element e of G

$$ad = T_e Ad : T_eG \to End(T_eG) \qquad X, Y \in T_eG \mapsto ad_X(Y) = [X, Y]$$

- For $G = GL_n(K)$ with $K = R$ or C

$$T_eG = M_n(K) \qquad X \in M_n(K), g \in G \quad Ad_g(X) = gXg^{-1}$$

$$X, Y \in M_n(K) \quad ad_X(Y) = (T_e Ad)_X(Y) = XY - YX = [X, Y]$$

  - Curve from $e = I_d = c(0)$ tangent to $X = c(1)$: $c(t) = \exp(tX)$

    and transform by Ad : $\gamma(t) = Ad \exp(tX)$

$$ad_X(Y) = (T_e Ad)_X(Y) = \frac{d}{dt}\gamma(t)Y\Big|_{t=0} = \frac{d}{dt}\exp(tX)Y\exp(tX)^{-1}\Big|_{t=0} = XY - YX$$

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Lie Group Adjoint Representation

- the adjoint representation of a Lie group Ad_g is a way of representing its elements as linear transformations of the Lie algebra, considered as a vector space

$$Ad_g = \left(d\Psi_g \right)_e : \mathfrak{g} \to \mathfrak{g}$$
$$X \mapsto Ad_g(X) = gXg^{-1}$$

$$\Psi : G \to Aut(G)$$
$$g \mapsto \Psi_g(h) = ghg^{-1}$$
$$ad = T_e Ad : T_e G \to End(T_e G)$$
$$X, Y \in T_e G \mapsto ad_X(Y) = [X, Y]$$

Lie Group Co-Adjoint Representation

- the coadjoint representation of a Lie group Ad_g^*, is the dual of the adjoint representation ($\mathfrak{g}^*$ denotes the dual space to $\mathfrak{g}$):

$$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*, \text{ then } \left\langle Ad_g^* F, Y \right\rangle = \left\langle F, Ad_{g^{-1}} Y \right\rangle$$

$$K = Ad_g^* = \left(Ad_{g^{-1}} \right)^* \quad \text{and} \quad K_*(X) = -\left(ad_X \right)^*$$

THALES

Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Co-adjoint Orbits as Homogeneous Symplectic Manifold by KKS 2-form

- A coadjoint orbit: $O_F = \left\{ Ad_g^* F, g \in G \right\}$ subset of $\mathfrak{g}^*, F \in \mathfrak{g}^*$

  carry a natural homogeneous symplectic structure by a closed G-invariant 2-form:

$$\sigma_\Omega\left(K_{*X}F, K_{*Y}F\right) = B_F\left(X,Y\right) = \left\langle F, [X,Y] \right\rangle, X, Y \in \mathfrak{g}$$

- The coadjoint action on O_F is a Hamiltonian G-action with moment map $\Omega \to \mathfrak{g}^*$

Souriau Foundamental Theorem « *Every symplectic manifold is a coadjoint orbit* » is based on classification of symplectic homogeneous Lie group actions by Souriau, Kostant and Kirillov

$g \in G$ → $O_F = \left\{ Ad_g^* F, g \in G, F \in \mathfrak{g}^* \right\}$ → $\sigma_\Omega\left(ad_F X, ad_F Y\right) = \left\langle F, [X,Y] \right\rangle$, $X, Y \in \mathfrak{g}, F \in \mathfrak{g}^*$

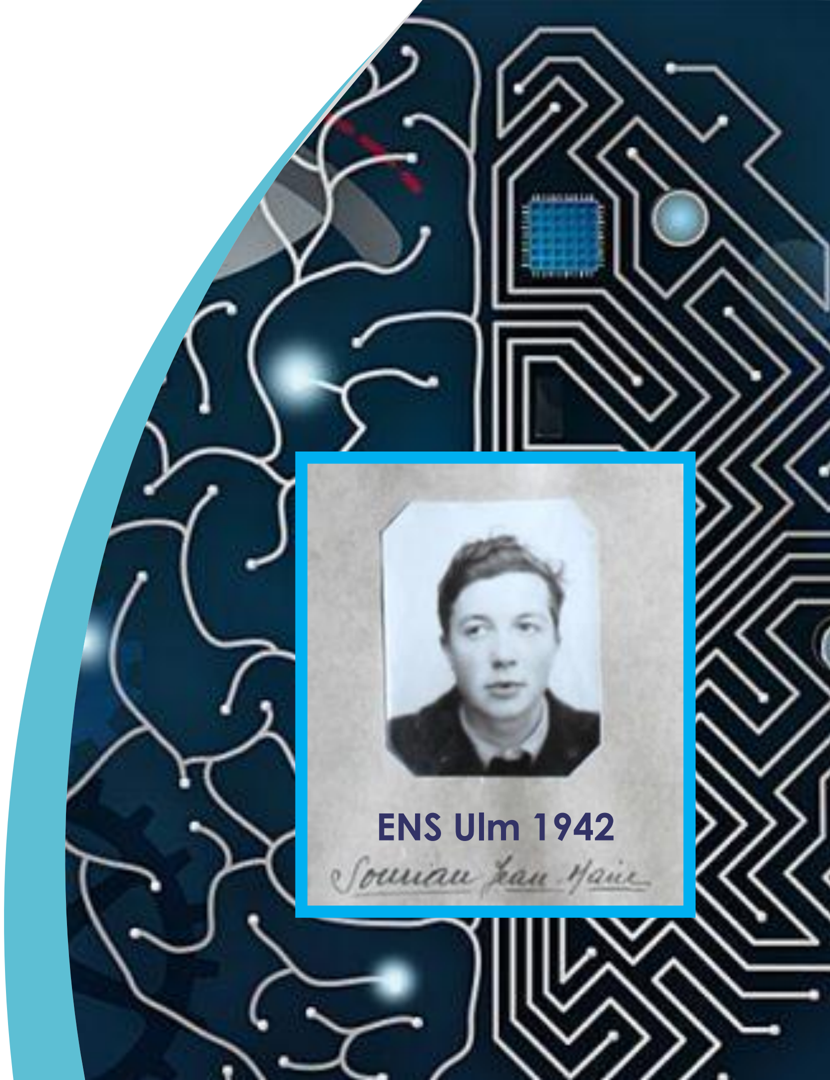
Lie Group

Coadjoint Orbit
(action of Lie Group on dual Lie algebra)

Homogeneous Symplectic Manifold
(a smooth manifold with a closed differential 2-form σ, such that dσ=0, where the Lie Group acts transitively)

THALES

Fisher-Koszul-Souriau Metric and Geometric Structures of Inference and Learning

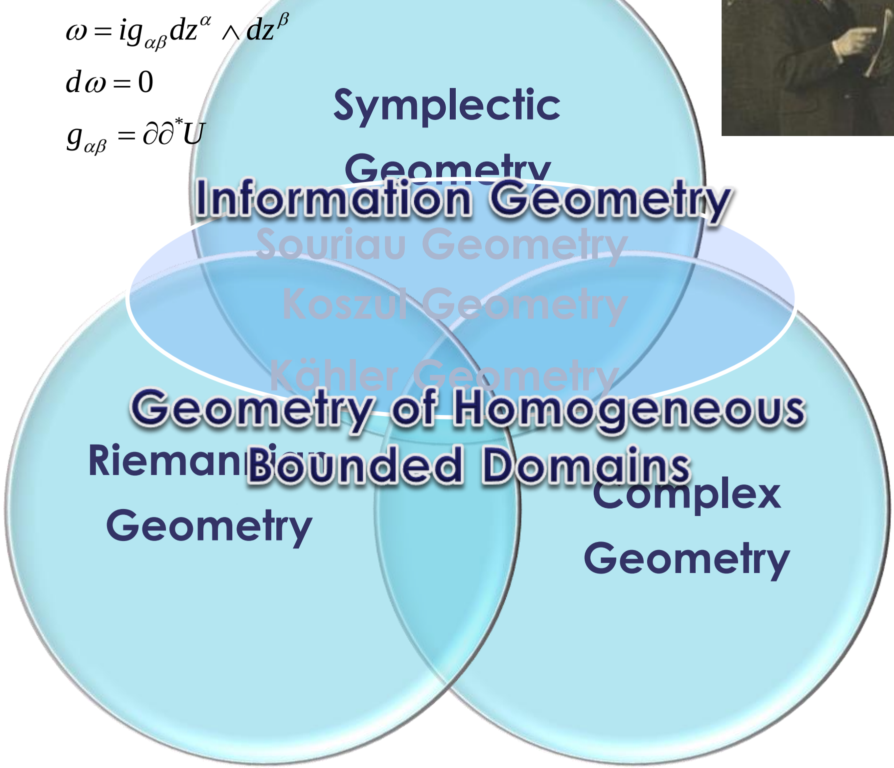


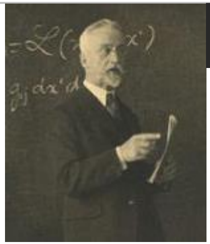
Elementary Structures of Information Geometry

$$\omega = i g_{\alpha\beta} dz^{\alpha} \wedge dz^{\beta}$$

$$d\omega = 0$$

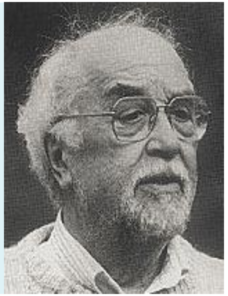
$$g_{\alpha\beta} = \partial\partial^{*} U$$





Seminal work of Elie Cartan

Geometry of Jean-Marie Souriau
Study of homogeneous symplectic manifolds geometry with the action of dynamical groups. Introduction of the Lagrange-Souriau 2-form and Lie Groups Thermodynamics.



Geometry of Jean-Louis Koszul
Study of homogeneous bounded domains geometry, symmetric homogeneous spaces and sharp convex cones. Introduction of an invariant 2-form.

Geometry of Erich Kähler
Study of differential manifolds geometry equipped with a unitary structure satisfying a condition of integrability. The homogeneous Kähler case studied by André Lichnerowicz.

Fisher Metric and Fréchet-Darmois (Cramer-Rao) Bound

- Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of the Fisher Information Matrix: $I(\theta)$

$$R_{\hat{\theta}} = E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^{+}\right] \geq I(\theta)^{-1} \qquad \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log p_\theta(z)}{\partial\theta_i \partial\theta_j^*}\right]$$

- Rao has proposed to introduced an invariant metric in parameter space of density of probabilities (axiomatised by N. Chentsov):

$$ds_\theta^2 = Kullback\_Divergence\left(p_\theta(z), p_{\theta+d\theta}(z)\right)$$

$$ds_\theta^2 = -\int p_\theta(z) \log \frac{p_{\theta+d\theta}(z)}{p_\theta(z)} dz$$

$$w = W(\theta) \Rightarrow ds_w^2 = ds_\theta^2$$

$$ds_\theta^2 \underset{Taylor}{\approx} \sum_{i,j} g_{ij} d\theta_i d\theta_j^* = \sum_{i,j} \left[I(\theta)\right]_{i,j} d\theta_i d\theta_j^* = d\theta^{+}.I(\theta).d\theta$$

Distance Between Gaussian Density with Fisher Metric

Fisher Matrix for Gaussian Densities:

$$I(\theta)=\begin{bmatrix}\frac{1}{\sigma^2} & 0\\ 0 & \frac{2}{\sigma^2}\end{bmatrix} \quad \text{avec} \quad E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^T\right]\geq I(\theta)^{-1} \quad \text{et} \quad \theta=\begin{pmatrix}m\\ \sigma\end{pmatrix}$$

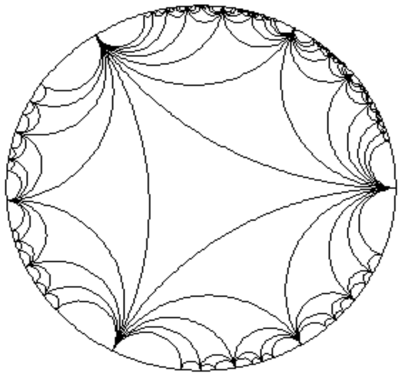
- Fisher matrix induced the following differential metric :

$$ds^2=d\theta^T.I(\theta).d\theta=\frac{dm^2}{\sigma^2}+2.\frac{d\sigma^2}{\sigma^2}=\frac{2}{\sigma^2}\left[\left(\frac{dm}{\sqrt{2}}\right)^2+(d\sigma)^2\right]$$

- Poincaré Model of upper half-plane and unit disk

$$z=\frac{m}{\sqrt{2}}+i.\sigma \qquad \omega=\frac{z-i}{z+i}\quad \left(|\omega|<1\right)$$

$$\Rightarrow ds^2=8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$



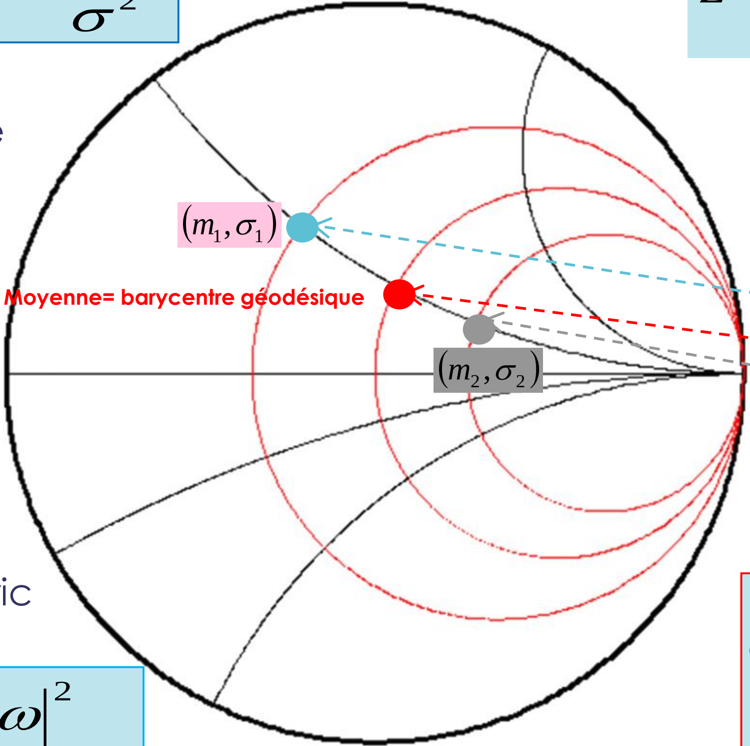
1 monovariate gaussian = 1 point in Poincaré unit disk

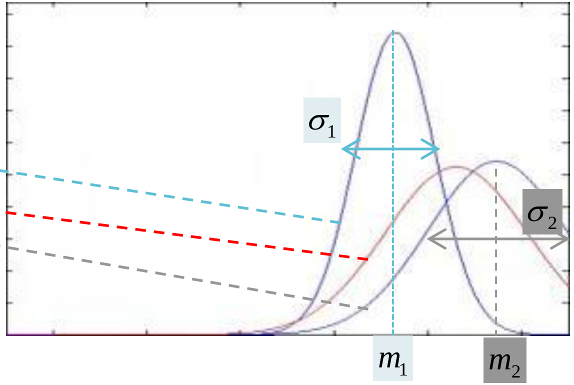
$$ds^2 = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2}$$

Fisher Metric in
Poincaré Half-Plane

$$z = \frac{m}{\sqrt{2}} + i.\sigma$$

$$\omega = \frac{z-i}{z+i} \quad \left(|\omega| < 1\right)$$





Poincaré-Fisher metric
In Unit Disk

$$ds^2 = 8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

$$d^2\left(\{m_1,\sigma_1\},\{m_2,\sigma_2\}\right) = 2.\left(\log\frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2$$

$$\text{with} \quad \delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)}-\omega^{(2)}}{1-\omega^{(1)}\omega^{(2)*}}\right|$$

Gradient descent for Learning

- Information geometry has been derived from invariant geometrical structure involved in statistical inference. The Fisher metric defines a Riemannian metric as the Hessian of two dual potential functions, linked to dually coupled affine connections in a manifold of probability distributions. With the Souriau model, this structure is extended preserving the Legendre transform between two dual potential function parametrized in Lie algebra of the group acting transitively on the homogeneous manifold.
- Classically, to optimize the parameter θ of a probabilistic model, based on a sequence of observations y_t , is an online gradient descent with learning rate η_t , and the loss function $l_t = -\log p\left(y_t / \hat{y}_t\right)$:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t \frac{\partial l_t\left(y_t\right)^T}{\partial \theta}$$

Information Geometry & Natural Gradient

- This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. **S.I. Amari** has introduced the **natural gradient** to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the **Fisher information matrix** with respect to parameter θ , given by:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

$$I(\theta) = [g_{ij}]$$

$$\text{with } g_{ij} = \left[-E_{y \approx p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} = \left[E_{y \approx p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial \theta_i} \frac{\partial \log p(y/\theta)}{\partial \theta_j} \right] \right]_{ij}$$

Natural Gradient & Stochastic Gradient: Natural Langevin Dynamics

Natural Langevin Dynamics: Natural Gradient with Langevin Stochastics descent

- To regularize solution and avoid over-fitting, Stochastic gradient is used, as Langevin Stochastic Gradients
- **Yann Ollivier** (FACEBOOK FAIR, previously CNRS LRI Orsay) and **Gaëtan Marceau-Caron** (MILA, previously CNRS LRI Orsay and THALES LAS/ATM & TRT PhD) have proposed to coupled **Natural Gradient** with **Langevin Dynamics**: **Natural Langevin Dynamics (Best SMF/SEE GSI'17 paper)**

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial\left(l_t(y_t)^T - \frac{1}{N}\log\alpha(\theta_{t-1})\right)}{\partial\theta} + \sqrt{\frac{2\eta_t}{N}} I(\theta_{t-1})^{-1/2} N(0, I_d)$$

- The resulting natural Langevin dynamics combines the advantages of Amari's natural gradient descent and Fisher-preconditioned Langevin dynamics for large neural networks

Dual Entropic Natural Gradient

- We can define a natural gradient with dual potential given by Shannon Entropy *H* (Legendre transform of characteristic fonction *G*, logarithm of partitiun function).

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t (y_t / \theta)^T}{\partial \theta}$$

$$\theta = \nabla S(\eta) = h(\eta) \qquad \eta = \nabla \Phi(\theta) = g(\theta) \qquad S(\eta) = \underset{\theta \in \Theta}{Sup} \{ \langle \theta, \eta \rangle - \Phi(\theta) \}$$

$$\eta_t \leftarrow \eta_{t-1} - \alpha_t \frac{\partial l_t \left(y_t / h(\eta_{t-1})\right)^T}{\partial \theta}$$

$$\nabla_\eta l_i \left(h(\eta)\right) = \nabla_\eta h(\eta) \nabla_\theta l_t \left(h(\eta)\right) \Rightarrow \nabla_\theta l_t \left(h(\eta)\right) = \left[\nabla_\eta h(\eta)\right]^{-1} \nabla_\eta l_i \left(h(\eta)\right)$$

$$\eta_t \leftarrow \eta_{t-1} - \alpha_t \left[\nabla^2 S(\eta_{t-1})\right]^{-1} \frac{\partial l_t \left(y_t / h(\eta_{t-1})\right)^T}{\partial \eta}$$

Natural Dual Entropic Gradient

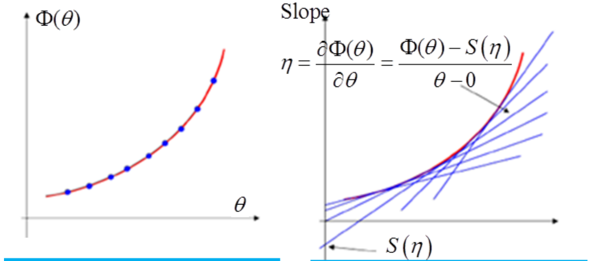
Legendre Transform, Dual Potentials & Fisher Metric

- **S.I. Amari** has proved that the Riemannian metric in an exponential family is the **Fisher information matrix** defined by:

$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j}\right]_{ij} \quad \text{with} \quad \Phi(\theta) = -\log \int_R e^{-\langle \theta, y \rangle} dy$$

- and the dual potential, the **Shannon entropy**, is given by the **Legendre transform**:

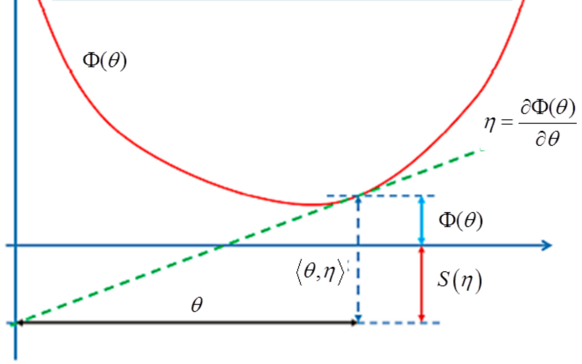
$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with} \quad \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$



Classical Geometry (curve is given by a continuum of points)

Plücker Geometry (curve is given by the envelop of its tangents)

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta)$$
$$\text{with} \quad \frac{\partial S(\eta)}{\partial \eta} = \theta \quad \text{and} \quad \frac{\partial \Phi(\theta)}{\partial \theta} = \eta$$



Fisher Metric and Koszul 2 form on sharp convex cones

Koszul-Vinberg Characteristic Function, Koszul Forms

- **J.L. Koszul** and **E. Vinberg** have introduced an affinely invariant Hessian metric on a sharp convex cone through its **characteristic function**

$$\Phi_\Omega(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y\rangle} dy = -\log \psi_\Omega(\theta) \quad \text{with } \theta \in \Omega \text{ sharp convex cone}$$

$$\psi_\Omega(\theta) = \int_{\Omega^*} e^{-\langle \theta, y\rangle} dy \quad \text{with Koszul-Vinberg Characteristic function}$$


Jean-Louis Koszul

- **1st Koszul form α**: $\alpha = d\Phi_\Omega(\theta) = -d\log\psi_\Omega(\theta)$
- **2nd Koszul form γ**: $\gamma = D\alpha = Dd\log\psi_\Omega(\theta)$

$$\left(Dd\log\psi_\Omega(x)\right)(u) = \frac{1}{\psi_\Omega(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi\right)^2\right] > 0 \quad \text{with} \quad F(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle} \quad \text{and} \quad G(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle}\langle u,\xi\rangle$$

- Diffeomorphism: $\eta = \alpha = -d\log\psi_\Omega(\theta) = \int_{\Omega^*} \xi p_\theta(\xi) d\xi \quad \text{with } p_\theta(\xi) = \dfrac{e^{-\langle \xi,\theta\rangle}}{\int_{\Omega^*} e^{-\langle \xi,\theta\rangle} d\xi}$
- Legendre transform: $S_\Omega(\eta) = \langle \theta,\eta\rangle - \Phi_\Omega(\theta) \quad \text{with} \quad \eta = d\Phi_\Omega(\theta) \quad \text{and} \quad \theta = dS_\Omega(\eta)$

Fisher Metric and Souriau 2-form: Lie Groups Thermodyamics

Statistical Mechanics, Dual Potentials & Fisher Metric

- In geometric statistical mechanics, **J.M. Souriau** has developed a "**Lie groups thermodynamics**" of dynamical systems where the (maximum entropy) **Gibbs density is covariant** with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda \qquad U : M \to \mathfrak{g}^*$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

Jean-Marie Souriau

- In the Souriau **Lie groups thermodynamics** model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group.

Fisher-Souriau Metric and its invariance

Statistical Mechanics & Invariant Souriau-Fisher Metric

- In Souriau's **Lie groups thermodynamics**, the invariance by re-parameterization in information geometry has been replaced by invariance with respect to the action of the group. When an element of the group g acts on the element $\beta \in \mathfrak{g}$ of the Lie algebra, given by adjoint operator Ad_g . Under the action of the group , $Ad_g(\beta)$, **the entropy** $S(Q)$ **and the Fisher metric** $I(\beta)$ **are invariant:**

$$\beta \in \mathfrak{g} \rightarrow Ad_g(\beta) \Rightarrow \begin{cases} S\left[Q\left(Ad_g(\beta)\right)\right] = S(Q) \\ I\left[Ad_g(\beta)\right] = I(\beta) \end{cases}$$

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

Fisher-Souriau Metric Definition by Souriau Cocycle & Moment Map

Statistical Mechanics & Fisher Metric

- Souriau has proposed a Riemannian metric that we have identified as a generalization of the Fisher metric:

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)$$

$$\text{with } \tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\left\langle Q,ad_{Z_1}(Z_2)\right\rangle \text{ where } ad_{Z_1}(Z_2)=[Z_1,Z_2]$$

- The tensor $\tilde{\Theta}$ used to define this extended Fisher metric is defined by the moment map $J(x)$, from M (homogeneous symplectic manifold) to the dual Lie algebra $\mathfrak{g}^*$, given by:

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\} \text{ with } J(x):M\to\mathfrak{g}^* \text{ such that } J_X(x)=\left\langle J(x),X\right\rangle,\ X\in\mathfrak{g}$$

- This tensor $\tilde{\Theta}$ is also defined in tangent space of the cocycle $\theta(g)\in\mathfrak{g}^*$ (this cocycle appears due to the non-equivariance of the coadjoint operator Ad_g^*, action of the group on the dual lie algebra): $Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$

$$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\to\Re \qquad \text{with } \Theta(X)=T_e\theta\left(X(e)\right)$$

$$X,Y\mapsto\left\langle\Theta(X),Y\right\rangle$$

Fisher-Souriau Metric as a non-null Cohomology extension of KKS 2 form (Kirillov-Kostant-Souriau 2 form)

Souriau definition of Fisher Metric is related to the extension of KKS 2-form (Kostant-Kirillov-Souriau) in case of non-null Cohomogy:

Souriau-Fisher Metric

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

$$\text{with } \tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\langle Q,[Z_1,Z_2]\rangle$$

Non-null cohomology: aditional term from Souriau Cocycle

Equivariant KKS 2 form

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_X,J_Y\} \text{ with } J(x):M\to\mathfrak{g}^* \text{ such that } J_X(x)=\langle J(x),X\rangle,\ X\in\mathfrak{g}$$

$$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\to\Re \qquad \text{with } \Theta(X)=T_e\theta(X(e))$$

$$X,Y\mapsto\langle\Theta(X),Y\rangle$$

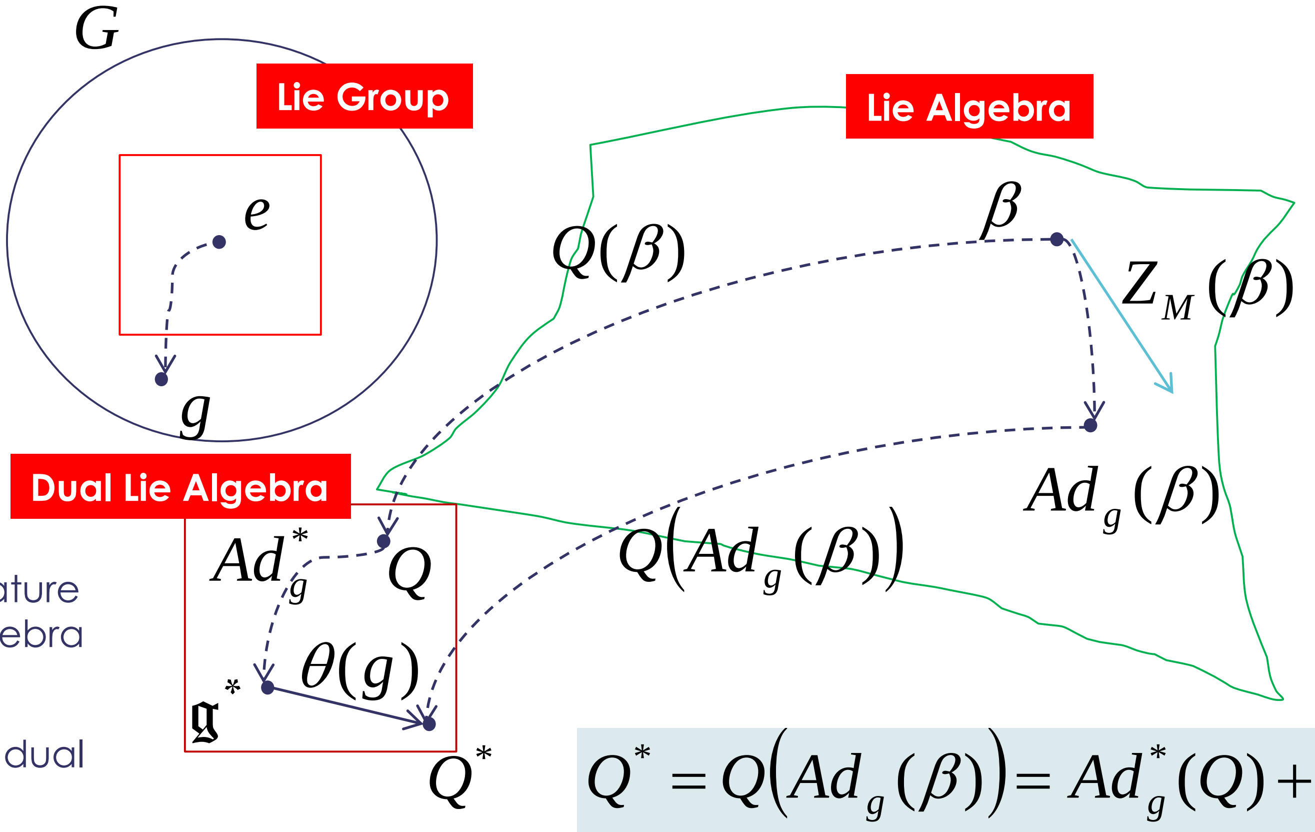
$$\tilde{\Theta}(\beta,Z)+\langle Q,[\beta,Z]\rangle=0$$

$$\beta\in Ker\,\tilde{\Theta}_\beta$$

Souriau Fundamental Equation of Lie Group Thermodynamics

$$Q(Ad_g(\beta))=Ad_g^*(Q)+\theta(g)$$

Fundamental Souriau Theorem



β : (Planck) température element of Lie algebra

Q : Heat, element of dual Lie Algebra

$$Q^* = Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$$

Non-equivariance of Coadjoint operator

- Non-equivariance of Coadjoint operator:

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta\left(g\right)$$

- This is the action of Lie Group on Moment map:

$$J\left(\Phi_g(x)\right) = a(g, J(x)) = Ad_g^*\left(J(x)\right) + \theta(g)$$

- By noting the action of the group on the dual space of the Lie algebra:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, (s,\xi) \mapsto s\xi = Ad_s^*\xi + \theta(s)$$

- Associativity is given by:

$$(s_1 s_2)\xi = Ad_{s_1 s_2}^*\xi + \theta(s_1 s_2) = Ad_{s_1}^* Ad_{s_2}^*\xi + \theta(s_1) + Ad_{s_1}^*\theta(s_2)$$

$$(s_1 s_2)\xi = Ad_{s_1}^*\left(Ad_{s_2}^*\xi + \theta(s_2)\right) + \theta(s_1) = s_1\left(s_2\xi\right) \quad , \quad \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^*$$

Les Houches 27th-31st July 2020
Joint Structures and Common Foundations of Statistical Physics,
Information Geometry and Inference for Learning (SPIGL'20)

Souriau Cocycle

- $\theta(g) \in \mathfrak{g}^*$ is called nonequivariance one-cocycle, and it is a measure of the lack of equivariance of the moment map.

$$\theta(st) = J((st).x) - Ad_{st}^* J(x)$$

$$\theta(st) = \left[J(s.(t.x)) - Ad_s^* J(t.x) \right] + \left[Ad_s^* J(t.x) - Ad_s^* Ad_t^* J(x) \right]$$

$$\theta(st) = \theta(s) + Ad_s^* \left[J(t.x) - Ad_t^* J(x) \right]$$

$$\theta(st) = \theta(s) + Ad_s^* \theta(t)$$

Souriau one-cocycle and compute 2-cocycle

$$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\rightarrow\Re \qquad \text{with } \Theta(X)=T_e\theta(X(e))$$

$$\mathrm{X,Y}\mapsto\langle\Theta(X),Y\rangle$$

- We can also compute tangent of one-cocycle θ at neutral element, to compute 2-cocycle Θ :

$$\zeta\in\mathfrak{g}\ ,\quad \theta_\zeta(s)=\langle\theta(s),\zeta\rangle=\langle J(s.x),\zeta\rangle-\langle Ad_s^*J(x),\zeta\rangle$$

$$\theta_\zeta(s)=\langle J(s.x),\zeta\rangle-\langle J(x),Ad_{s^{-1}}\zeta\rangle$$

$$T_e\theta_\zeta(\xi)=\langle T_xJ.\xi_p(x),\zeta\rangle+\langle J(x),ad_\xi\zeta\rangle \quad \text{with } \xi_p=X_{\langle J,\xi\rangle}$$

$$T_e\theta_\zeta(\xi)=X_{\langle J(x),\xi\rangle}\left[\langle J(x),\zeta\rangle\right]+\langle J(x),[\xi,\zeta]\rangle$$

$$T_e\theta_\zeta(\xi)=-\{\langle J,\xi\rangle,\langle J,\zeta\rangle\}+\langle J(x),[\xi,\zeta]\rangle=\Theta(\xi)$$

Souriau Tensor

$$\tilde{\Theta}(X,Y) = J_{[X,Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y\rangle \quad , \quad X,Y \in \mathfrak{g}$$

$$\tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([X,Y],Z) = 0 \quad , \quad X,Y,Z \in \mathrm{g}$$

- By differentiating the equation on affine action, we have:

$$T_x J(\xi_p(x)) = -ad_\xi^* J(x) + \Theta(\xi,.)$$

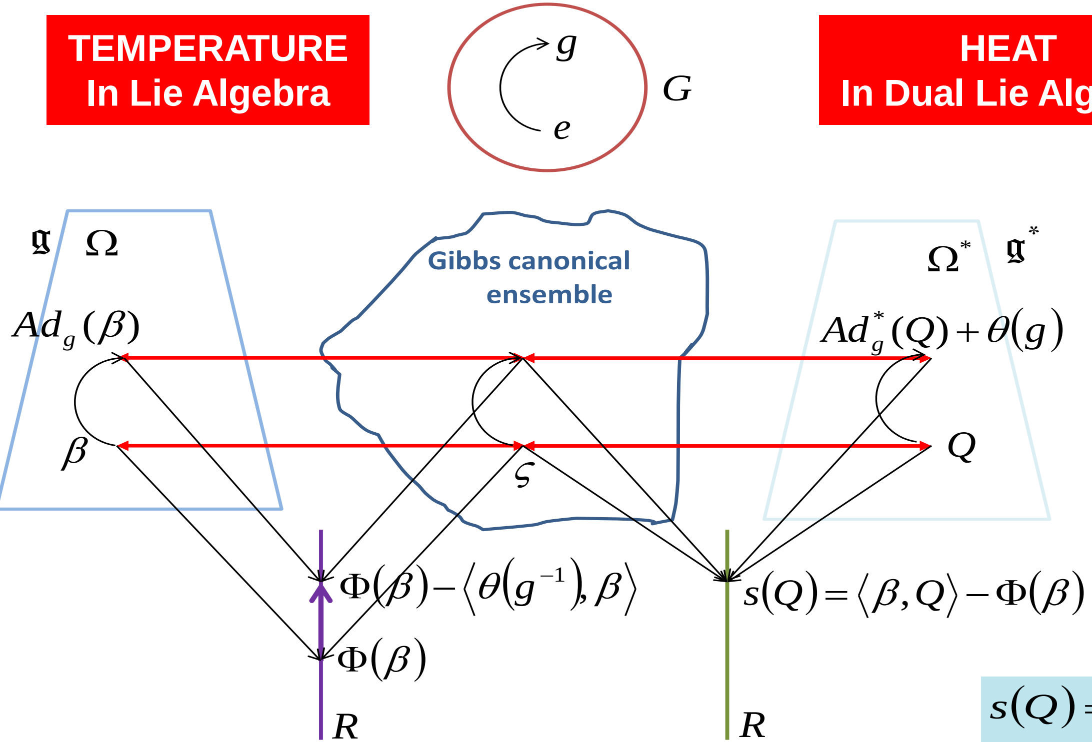
$$dJ(Xx) = ad_X J(x) + d\theta(X) \quad , \quad x \in M, X \in \mathfrak{g}$$

$$\langle dJ(Xx), Y\rangle = \langle ad_X J(x), Y\rangle + \langle \mathrm{d}\theta(X), Y\rangle, \quad x \in M, X, Y \in \mathfrak{g}$$

$$\langle dJ(Xx), Y\rangle = \langle J(x), [X,Y]\rangle + \langle \mathrm{d}\theta(X), Y\rangle \quad = \{\langle J, X\rangle, \langle J, Y\rangle\}(x)$$

$$\langle J(x), [X,Y]\rangle - \{\langle J, X\rangle, \langle J, Y\rangle\}(x) = -\langle \mathrm{d}\theta(X), Y\rangle$$

Souriau-Fisher Metric & Souriau Lie Groups Thermodynamics: Bedrock for Lie Group Machine Learning



Logarithm of Partition Function (Massieu Characteristic Function)

Entropy

Fisher Metric

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) \geq 0$$

$$I(\beta) = I\left(Ad_g(\beta)\right) = -\frac{\partial^2\Phi}{\partial\beta^2} = \frac{\partial^2 \log \int_M e^{-\langle\beta,U(\xi)\rangle} d\lambda}{\partial\beta^2}$$

Entropy invariant under the action of the group

Legendre Clairaut

$$s(Q) = \langle\beta,Q\rangle - \Phi(\beta) = \langle\Theta^{-1}(Q),Q\rangle - \Phi\left(\Theta^{-1}(Q)\right)$$

$$Q = \Theta(\beta) = \frac{\partial\Phi}{\partial\beta} \in \mathfrak{g}^* \qquad \beta = \Theta^{-1}(Q) \in \mathfrak{g}$$

Link with Classical Thermodynamics

We have the reciprocal formula:

$Q = \dfrac{\partial \Phi}{\partial \beta}$	$\beta = \dfrac{\partial s}{\partial Q}$
$s(Q) = \left\langle \dfrac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi$	$\Phi(\beta) = \left\langle Q, \dfrac{\partial s}{\partial Q} \right\rangle - s$

For Classical Thermodynamics (Time translation only), we recover the definition of Boltzmann-Clausius Entropy:

$$\begin{cases} \beta = \dfrac{\partial s}{\partial Q} \\ \beta = \dfrac{1}{T} \end{cases} \Rightarrow ds = \frac{dQ}{T}$$

Souriau Model of Covariant Gibbs Density

Covariant Souriau-Gibbs density

- Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$p_{Gibbs}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \beta \rangle} = \frac{e^{-\langle U(\xi), \beta \rangle}}{\int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}$$

$$\text{with } \Phi(\beta) = -\log \int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int_M U(\xi) e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}{\int_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega} = \int_M U(\xi) p(\xi) d\lambda_\omega$$

- We can express the Gibbs density with respect to Q by inverting the relation $Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta)$. Then $p_{Gibbs,Q}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \Theta^{-1}(Q) \rangle}$ with $\beta = \Theta^{-1}(Q)$

Souriau Lie Groups Thermodynamics: Geometric Calorific Capacity

Nous prenons désormais Z dans C. La valeur moyenne du moment $\Psi(x)$ dans l'état de Gibbs est égal à la dérivée

$$Q = z'(Z);$$

$Z \mapsto Q$ est un difféomorphisme analytique de C sur un ouvert convexe de $\mathfrak{g}^*$; la transformée de Legendre s de z :

$$s(Q) = QZ - z$$

y est convexe et vérifie $Z = s'(Q)$; la dérivée seconde:

$$\boxed{K = z''(Z)}$$

est un tenseur positif, dont l'inverse est égal à $s''(Q)$.

K munit l'ensemble C d'une structure riemannienne invariante par l'action du groupe; pour cette structure, l'application linéaire $\mathrm{Ad}(Z)$ est antihermitienne.

L'application f_Z, définie par:

$$\boxed{f_Z(Z', Z'') = K([Z, Z'], Z'') \quad \forall Z', Z'' \in \mathfrak{g}}$$

est un cocycle symplectique, cohomologue à f [formule (2,7 C)]; son noyau est l'orthogonal de l'orbite adjointe de Z pour la structure riemannienne de C.

Souriau-Fisher Metric is a Geometrization of Thermodynamical «Calorific Capacity» (Pierre Duhem has deeply developed this idea of « generalized capacities »)

■ Dans le cas classique, on ne considère que le groupe de dimension 1 des translations temporelles (qui n'est défini qu'après avoir choisi un référentiel - par exemple celui de la boîte qui contient le gaz). Alors, avec des unités convenables, Z est l'inverse de la TEMPERATURE ABSOLUE; z est le POTENTIEL THERMODYNAMIQUE DE PLANCK; $-s$ est l'ENTROPIE; Q est l'ENERGIE INTERNE; K caractérise la CAPACITE CALORIFIQUE. ■

THALES

Souriau Lie Group Thermodynamics: Geometric Calorific Capacity

Il faut bien entendu que cette intégrale soit convergente ; nous définirons l'ensemble canonique de Gibbs Ω comme le plus grand ouvert (dans l'algèbre de Lie) où cette intégrale est localement normalement convergente (en Θ). On montre que Ω est convexe, et que z est une fonction C^∞ sur Ω ; que la dérivée $Q = \frac{\partial z}{\partial \Theta}$ coïncide avec la valeur moyenne de l'énergie E (Q généralise donc la chaleur) ; que le tenseur $\frac{\partial Q}{\partial \Theta}$ est symétrique et positif (il généralise la capacité calorifique). Il en résulte que z est fonction convexe de Θ ; la transformation de Legendre lui associe une fonction concave, à savoir

$$Q \mapsto s = z - Q\Theta \qquad (7.3)$$

s est l'entropie.

Souriau-Fisher Metric based on cocycle

pour chaque "température" Θ, définissons un tenseur f_Θ, somme du cocycle f (défini en (3.2)) et du cobord de la chaleur :

(7.4) $$f_\Theta(Z,Z') = f(Z,Z') + Q[Z,Z']$$

f_Θ jouit alors des propriétés suivantes :

5)
a) f_Θ est un cocycle symplectique ;

b) $\Theta \in \ker f_\Theta$

c) Le tenseur symétrique g_Θ, défini sur l'ensemble de valeurs de $ad(\Theta)$ par

$$g_\Theta([\Theta,Z],[\Theta,Z']) = f_\Theta(Z,[\Theta,Z'])$$

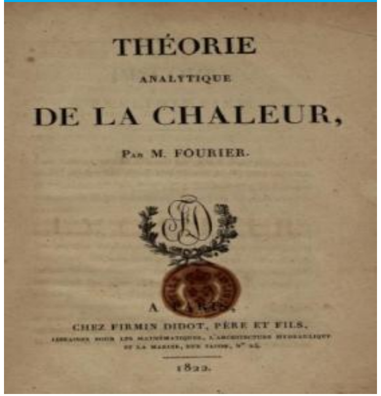
est positif (et même défini positif si l'action du groupe est effective).

Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G, son cocycle symplectique f et les couples Θ, Q. Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

Souriau Lie Groups Thermodynamics: General Equations

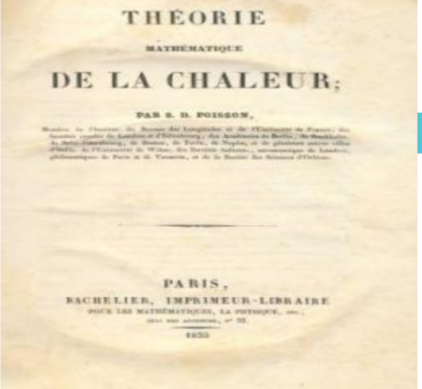
Ces formules sont universelles, en ce sens qu'elles ne mettent pas en jeu la variété symplectique U - mais seulement le groupe G, son cocycle symplectique f et les couples Θ, Q. Peut-être cette "thermodynamique des groupes de Lie" a-t-elle un intérêt mathématique.

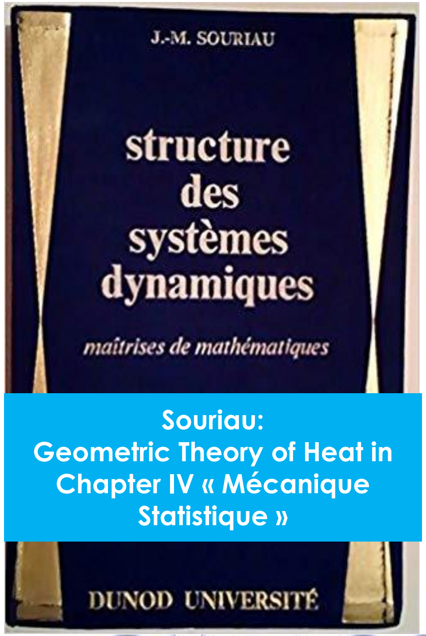
Fourier: Analytical Theory of Heat



Clausius: Mechanical Theory of Heat

Poisson: Mathematical Theory of Heat





Souriau: Geometric Theory of Heat in Chapter IV « Mécanique Statistique »

SOURIAU GEOMETRIC THEORY OF HEAT

Multivariate Gaussian Density as 1st order Maximum Entropy in Souriau Book (Chapter IV)

Exemple : (*loi normale*) :

Prenons le cas $V = R^n$, $\lambda =$ mesure de Lebesgue, $\Psi(x) \equiv \begin{pmatrix} x \\ x \otimes x \end{pmatrix}$;

un élément Z du dual de E peut se définir par la formule

$$Z(\Psi(x)) \equiv \bar{a}.x + \tfrac{1}{2}\bar{x}.H.x$$

[$a \in R^n$; H = matrice symétrique]. On vérifie que la convergence de l'intégrale I_0 a lieu si la matrice H est positive (1) ; dans ce cas la loi de Gibbs s'appelle *loi normale de Gauss* ; on calcule facilement I_0 en faisant le changement de variable $x^* = H^{1/2}x + H^{-1/2}a$ (2) ; il vient

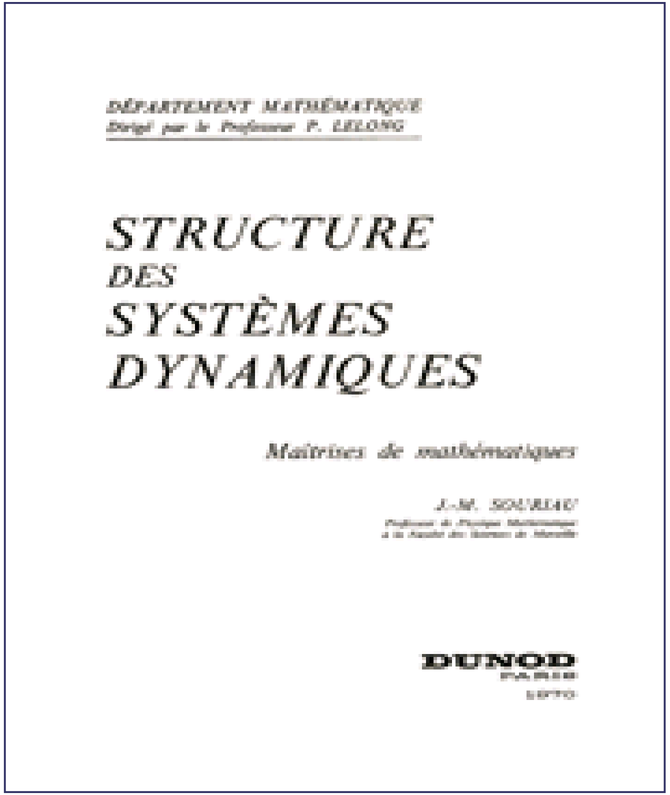
$$\boxed{z = \tfrac{1}{2}\left[\bar{a}.H^{-1}.a - \log(\text{dét}(H)) + n\log(2\pi)\right]}$$

alors la convergence de I_1 a lieu également ; on peut donc calculer M, qui est défini par les moments du premier et du second ordre de la loi (16.196) ; le calcul montre que le moment du premier ordre est égal à $-H^{-1}.a$ et que les composantes du tenseur *variance* (16.196) sont égales aux éléments de la matrice H^{-1} ; le moment du second ordre s'en déduit immédiatement.

La formule (16.200) donne l'*entropie* :

$$\boxed{s = \frac{n}{2}\log(2\pi e) - \frac{1}{2}\log(\text{dét}(H))} \; ;$$

(1) Voir *Calcul linéaire*, tome II.
(2) C'est-à-dire en recherchant l'*image* de la loi par l'application $x \mapsto x^*$.


DÉPARTEMENT MATHÉMATIQUE

STRUCTURE DES SYSTÈMES DYNAMIQUES

Maîtrises de mathématiques

J.-M. SOURIAU

DUNOD

http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Example of Multivariate Gaussian Law (real case)

Multivariate Gaussian law parameterized by moments

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2}} e^{-\frac{1}{2}(z-m)^T R^{-1}(z-m)}$$

$$\frac{1}{2}(z-m)^T R^{-1}(z-m) = \frac{1}{2}\left[z^T R^{-1} z - m^T R^{-1} z - z^T R^{-1} m + m^T R^{-1} m\right]$$

$$= \frac{1}{2} z^T R^{-1} z - m^T R^{-1} z + \frac{1}{2} m^T R^{-1} m$$

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2} e^{\frac{1}{2}m^T R^{-1} m}} e^{-\left[-m^T R^{-1} z + \frac{1}{2} z^T R^{-1} z\right]} = \frac{1}{Z} e^{-\langle \xi, \beta \rangle}$$

Gaussian Density is a 1st order Maximum Entropy Density !

$$\xi = \begin{bmatrix} z \\ zz^T \end{bmatrix} \text{ and } \beta = \begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} = \begin{bmatrix} a \\ H \end{bmatrix} \text{ with } \langle \xi, \beta \rangle = a^T z + z^T H z = Tr\left[z a^T + H^T z z^T\right]$$

THALES

SOURIAU ENTROPY AS INVARIANT CASIMIR FUNCTION IN COADJOINT REPRESENTATION

Gromov question: Are there « entropies » associated to moment maps

Bernoulli Lecture - What is Probability?

- 27 March 2018 - CIB - EPFL - Switzerland
- Lecturer: Mikhail Gromov
- https://bernoulli.epfl.ch/images/website/What_is_Probability_v2(2).mp4
- http://forum.cs-dc.org/uploads/files/1525172771489-alternative-probabilities-2018.pdf

Fisher Metric. Recall (Archimedes, 287-212 BCE) the *real moment map* from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ to the probability simplex $\triangle^n \subset \mathbb{R}^{n+1}$ for

$$(x_0, ..., x_n) \mapsto (p_0 = x_0^2, ..., p_n = x_n^2)$$

and observe following R. Fisher that the spherical metric (with constant curvature +1) thus transported to $\triangle^n$, call it ds^2 on $\triangle^n$, is equal, up to a scalar multiple, to the *Hessian of the entropy*

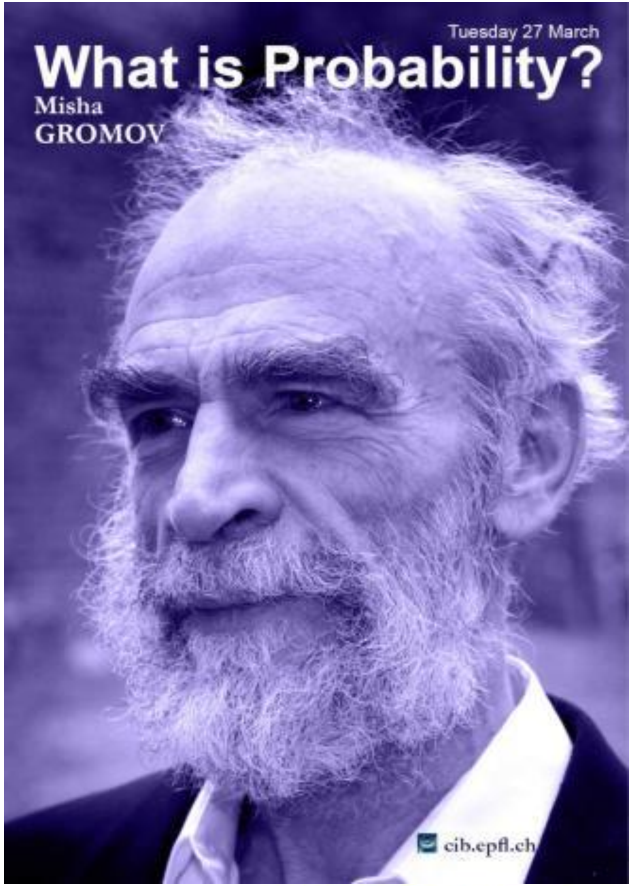
$ent\{p_0, ..., p_n\} = -\sum_i p_i \log p_i$.

$$ds^2 = const \frac{\partial^2 ent(p_i)}{dp_i dp_j}.$$

If, accordingly, we take the "inverse Hessian" – a kind of double integral "$\int\int ds^2$" for the *definition* of entropy – we arrive at

Question 2. Are there *interesting* "entropies" associated to (real and complex) moment maps of general toric varieties? Is there a *meaningful* concept of "generalised probability" grounded in positivity encountered in algebraic geometry?

****



OPEN

THALES

Souriau Entropy Invariance

Casimir Invariant Function in coadjoint representation

- We conclude the paper by a deeper study of Souriau model structure. We observe that Souriau Entropy $S(Q)$ defined on coadjoint orbit of the group has a property of invariance :

$$S\left(Ad_g^{\#}(Q)\right) = S(Q)$$

- with respect to Souriau affine definition of coadjoint action:

$$Ad_g^{\#}(Q) = Ad_g^{*}(Q) + \theta(g)$$

- where $\theta(g)$ is called the Souriau cocyle.

$$Q\left(Ad_g(\beta)\right) = Ad_g^{*}(Q) + \theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right) = S(Q)$$

New Entropy Definition: Function in Coadjoint Representation Invariant under the action of the Group

Hendrik Casimir (Thesis supervised by Niels Bohr & Paul Ehrenfest)

H.B.G. Casimir, On the Rotation of a Rigid Body in Quantum Mechanics, Doctoral Thesis, Leiden, 1931.

Entropy as Invariant Casimir Function in Coadjoint Representation

NEW GEOMETRIC DEFINITION OF ENTROPY

$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle=-C_{ij}^{k}Q_{k}\frac{\partial S}{\partial Q_{i}}\cdot\frac{\partial H}{\partial Q_{j}}$$

$$\left[e_i,e_j\right]=C_{ij}^{k}e_k \quad , \quad C_{ij}^{k} \quad \text{structure coefficients}$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0 \quad , \quad \forall H:\mathfrak{g}^{*}\rightarrow R,\ \ Q\in\mathfrak{g}^{*}$$

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_X,J_Y\} \quad \text{where} \quad J_X(x)=\langle J(x),X\rangle$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle \quad \text{with} \quad \Theta(X)=T_e\theta(X(e))$$

Souriau Entropy and Casimir Invariant Function

Geometric Definition of Entropy

- In the framework of Souriau Lie groups Thermodynamics, we can characterize the Entropy as a generalized Casimir invariant function in coadjoint representation,

Geometric Definition of Massieu Characteristic Function

- Massieu characteristic function (or log-partition function), dual of Entropy by Legendre transform, as a generalized Casimir function in adjoint representation.

Casimir Function Definition

- When M is a Poisson manifold, a function on M is a Casimir function if and only if this function is constant on each symplectic leaf (the non-empty open subsets of the symplectic leaves are the smallest embedded manifolds of M which are Poisson submanifolds)

Entropy Invariance under the action of the Group (1/2)

$$\beta \in \mathfrak{g} \rightarrow Ad_g(\beta) \Rightarrow \Psi(Ad_g(\beta)) = \int_M e^{-\langle U, Ad_g(\beta) \rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = \int_M e^{-\left\langle Ad^*_{g^{-1}} U, \beta \right\rangle} d\lambda_\omega = \int_M e^{-\left\langle U\left(Ad_{g^{-1}}\beta\right) - \theta\left(g^{-1}\right), \beta \right\rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = e^{\left\langle \theta\left(g^{-1}\right), \beta \right\rangle} \Psi(\beta)$$

$$\theta\left(g^{-1}\right) = -Ad^*_{g^{-1}}\theta(g) \Rightarrow \Psi(Ad_g(\beta)) = e^{-\left\langle Ad^*_{g^{-1}}\theta(g), \beta \right\rangle} \Psi(\beta)$$

$$\Phi(\beta) = -\log \Psi(\beta)$$

$$\Rightarrow \Phi\left(Ad_g(\beta)\right) = \Phi(\beta) - \left\langle \theta\left(g^{-1}\right), \beta \right\rangle = \Phi(\beta) + \left\langle Ad^*_{g^{-1}}\theta(g), \beta \right\rangle$$

Entropy Invariance under the action of the Group (2/2)

$$S(Q)=\langle Q,\beta\rangle-\Phi(\beta)\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Q\left(Ad_g\beta\right),Ad_g\beta\right\rangle-\Phi\left(Ad_g\beta\right)$$

$$\begin{cases} Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta\left(g\right) \\ \Phi\left(Ad_g(\beta)\right)=-\log\Psi(Ad_g(\beta))=-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle+\Phi\left(\beta\right) \end{cases}$$

$$\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g^*(Q)+\theta\left(g\right),Ad_g\beta\right\rangle+\left\langle\theta\left(g^{-1}\right),\beta\right\rangle-\Phi\left(\beta\right)$$

$$\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g^*(Q)+\theta\left(g\right),Ad_g\beta\right\rangle-\left\langle Ad_{g^{-1}}^*\theta(g),\beta\right\rangle-\Phi\left(\beta\right)$$

$$\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_{g^{-1}}^*Ad_g^*(Q)+Ad_{g^{-1}}^*\theta\left(g\right),\beta\right\rangle-\left\langle Ad_{g^{-1}}^*\theta(g),\beta\right\rangle-\Phi\left(\beta\right)$$

$$Ad_{g^{-1}}^*Ad_g^*(Q)=Q\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\langle Q,\beta\rangle-\Phi\left(\beta\right)=S\left(\beta\right)$$

Souriau Entropy and Casimir Invariant Function

Casimir Function and Entropy

- Classically, the Entropy is defined axiomatically as Shannon or von Neumann Entropies without any geometric structures constraints.
- Entropy could be built by Casimir Function Equation:

$$\left(ad^{*}_{\frac{\partial S}{\partial Q}} Q \right)_j + \Theta\left(\frac{\partial S}{\partial Q} \right)_j = C_{ij}^{k} ad^{*}_{\left(\frac{\partial S}{\partial Q}\right)^i} Q_k + \Theta_j = 0$$

$$\tilde{\Theta}(X,Y) = \langle \Theta(X), Y \rangle = J_{[X,Y]} - \{ J_X, J_Y \} = -\langle d\theta(X), Y \rangle \quad , \; X,Y \in \mathfrak{g}$$

$$\Theta(X) = T_e\theta(X(e))$$

$$\theta(g) = Q\left(Ad_g(\beta)\right) - Ad_g^{*}(Q)$$

Souriau Entropy and Casimir Invariant Function

Demo

- if we consider the heat expression $Q=\frac{\partial\Phi}{\partial\beta}$, that we can write $\delta\Phi-\langle Q,\delta\beta\rangle=0$.
- For each $\delta\beta$ tangent to the orbit, and so generated by an element Z of the Lie algebra, if we consider the relation $\Phi\left(Ad_g(\beta)\right)=\Phi(\beta)-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle$, we differentiate it at $g=e$ using the property that:

$$\tilde{\Theta}(X,Y)=-\langle d\theta(X),Y\rangle \ , \ X,Y\in\mathfrak{g}$$

- we obtain : $\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$
- From last Souriau equation, if we use the identities $\beta=\frac{\partial S}{\partial Q}$, $ad_\beta Z=[\beta,Z]$ and $\tilde{\Theta}(\beta,Z)=\langle\Theta(\beta),Z\rangle$
- Then we can deduce that: $\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),Z\right\rangle=0,\ \ \forall Z$
- So, Entropy $S(Q)$ should verify:

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0 \qquad \{S,H\}_{\tilde{\Theta}}(Q)=0 \ \ \forall H:\mathfrak{g}^*\to R,\ \ Q\in\mathfrak{g}^*$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

Souriau Entropy and Casimir Invariant Function

Demo

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,ad_{\frac{\partial S}{\partial Q}}\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q,\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\forall H,\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\Rightarrow ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau Entropy and Casimir Invariant Function

Link with Souriau development

- Souriau property: $\beta \in Ker\tilde{\Theta}_\beta \Rightarrow \langle Q,[\beta,Z]\rangle + \tilde{\Theta}(\beta,Z) = 0$

$$\Rightarrow \langle Q, ad_\beta Z\rangle + \tilde{\Theta}(\beta,Z) = 0 \Rightarrow \langle ad_\beta^* Q, Z\rangle + \tilde{\Theta}(\beta,Z) = 0$$

$$\beta = \frac{\partial S}{\partial Q} \Rightarrow \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q, Z \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, Z\right) = \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0, \forall Z$$

$$\Rightarrow ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

Souriau Entropy and Casimir Invariant Function

Dynamic equation

- The dual space of the Lie algebra foliates into coadjoint orbits that are also the level sets on the entropy.
- The information manifold foliates into level sets of the entropy that could be interpreted in the framework of Thermodynamics by the fact that motion remaining on this complex surfaces is non-dissipative, whereas motion transversal to these surfaces is dissipative, where the dynamic is given by:

$$\frac{dQ}{dt}=\left\{Q,H\right\}_{\tilde{\Theta}}=ad^{*}_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)$$

- with stable equilibrium when:

$$H=S\Rightarrow\frac{dQ}{dt}=\left\{Q,S\right\}_{\tilde{\Theta}}=ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Entropy Production and 2nd Principle

2nd Principle

- We can observe that: $dS = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q},\beta\right)dt$
- Where:

$$\tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q},\beta\right) = \tilde{\Theta}\left(\frac{\partial H}{\partial Q},\beta\right) + \left\langle Q,\left[\frac{\partial H}{\partial Q},\beta\right]\right\rangle$$

- showing that Entropy production is linked with Souriau tensor related to Fisher metric: $\frac{dS}{dt} = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q},\beta\right) \geq 0$
- It allows to introduce the stochastic extension based on a Stratonovich differential equation for the stochastic process given by the following relation by mean of Souriau's symplectic cocycle

$$dQ + \left[ad^*_{\frac{\partial H}{\partial Q}}Q + \Theta\left(\frac{\partial H}{\partial Q}\right)\right]dt + \sum_{i=1}^{N}\left[ad^*_{\frac{\partial H_i}{\partial Q}}Q + \Theta\left(\frac{\partial H_i}{\partial Q}\right)\right] \circ dW_i(t) = 0$$

Entropy Production and 2nd Principle

Demo

$$S(Q) = \langle Q, \beta \rangle - \Phi(\beta) \quad \text{with} \quad \frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$\frac{dS}{dt} = \left\langle Q, \frac{d\beta}{dt} \right\rangle + \left\langle ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right), \beta \right\rangle - \frac{d\Phi}{dt} = \left\langle Q, \frac{d\beta}{dt} \right\rangle + \left\langle ad^*_{\frac{\partial H}{\partial Q}} Q, \beta \right\rangle + \left\langle \Theta\left(\frac{\partial H}{\partial Q}\right), \beta \right\rangle - \frac{d\Phi}{dt}$$

$$\frac{dS}{dt} = \left\langle Q, \frac{d\beta}{dt} \right\rangle + \left\langle Q, \left[\frac{\partial H}{\partial Q}, \beta\right] \right\rangle + \tilde{\Theta}\left(\frac{\partial H}{\partial Q}, \beta\right) - \frac{d\Phi}{dt} = \left\langle Q, \frac{d\beta}{dt} \right\rangle + \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) - \left\langle \frac{\partial \Phi}{\partial \beta}, \frac{d\beta}{dt} \right\rangle$$

$$\frac{dS}{dt} = \left\langle Q, \frac{d\beta}{dt} \right\rangle + \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) - \left\langle \frac{\partial \Phi}{\partial \beta}, \frac{d\beta}{dt} \right\rangle \quad \text{with} \quad \frac{\partial \Phi}{\partial \beta} = Q$$

$$\frac{dS}{dt} = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) \geq 0, \forall H \quad \text{(link to positivity of Fisher metric)}$$

$$\text{if } H = S \underset{\frac{\partial S}{\partial Q} = Q}{\Rightarrow} \frac{dS}{dt} = \tilde{\Theta}_\beta(\beta, \beta) = 0 \text{ because } \beta \in Ker \tilde{\Theta}_\beta$$

THALES

Souriau Lie Groups Thermodynamics & Covariant Gibbs Density

Geometric (Planck) Temperature in the Lie Algrbra

- Let a Group G of a Manifold M with a moment map E, the **Geometric (Planck) Temperature** β is all elements of Lie Agebra $\mathfrak{g}$ of G such that the following integrals converges in a neighborhood of β : $I_0(\beta)=\int_M e^{-\langle\beta,U\rangle}d\lambda$
  - $\langle\beta,U\rangle$ notes the duality of $\mathfrak{g}$ and $\mathfrak{g}^*$
  - $d\lambda$ is the Liouville density on M

- <u>Theorem:</u> The function I_0 is infinitly differentiable C^∞ in Ω (the largest open proper subset of $\mathfrak{g}$) and is nth derivative for all $\beta\in\Omega$, the tensor integral is convergent: $I_n(\beta)=\int_M e^{-\langle\beta,U\rangle}U^{\otimes^n}d\lambda$

- To each temperature β, we can associate probability law on M with distribution function (such that the probability law has a mass equal to 1):

$$e^{\Phi(\beta)-\langle\beta,U(\xi)\rangle} \text{ with } \Phi(\beta)=-\log(I_0)=-\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda \text{ and } Q(\beta)=\int_M e^{\Phi(\beta)-\langle\beta,U(\xi)\rangle}Ud\lambda=\frac{I_1}{I_0}$$

  - The set of these probalities law is **Gibbs Ensemble of the Dynamic Group**, Φ is the **Thermodynamic Potential** and Q is the **Geometric Heat** $Q\in\mathfrak{g}^*$

Geometric Fisher Metric: Geometric Heat Capacity

- We can observe that the Geometric Heat Q is C^{∞} function of Geometric Temperature β in Dual Lie Algebra $\mathfrak{g}^{*}$:

$$\beta \in \mathfrak{g} \mapsto Q \in \mathfrak{g}^{*}$$

$$Q(\beta) = \int_M e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle} U d\lambda = \frac{I_1}{I_0}$$

$$\Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda$$

- We have: $Q = \dfrac{\partial \Phi}{\partial \beta}$

- Its derivative is a 2nd order symmetric tensor: $\dfrac{\partial Q}{\partial \beta} = \dfrac{I_2}{I_0} - \dfrac{I_1 \otimes I_1}{I_0} = \dfrac{I_2}{I_1} - Q \otimes Q$

$$-\frac{\partial Q}{\partial \beta} = \int_M e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle} [U - Q] \otimes [U - Q] d\lambda$$

$$-\frac{\partial Q}{\partial \beta} \geq 0 \qquad -\frac{\partial Q}{\partial \beta} = -\frac{\partial^2 \Phi}{\partial \beta^2}$$

- This quatratic form is positive, and positive definite for each $x \in M$ unless there exist a non null element $Z \in \mathfrak{g}$ such that $\langle U - Q, Z \rangle = 0$ (means that the moment U varies in an affine sub-manifold of $\mathfrak{g}^{*}$)

Distribution of probability by Group action

- The distribution density under the action of the Lie Group is given by:

$$\mu^* : e^{\Phi^* - \langle \beta^*, U \rangle}$$

$$\Phi^* = \Phi(\beta^*) = \Phi - \left\langle \theta(g^{-1}), \beta \right\rangle$$
$$\Phi^* = \Phi + \left\langle \theta(g), Ad_g \beta \right\rangle$$

(**)

$$\beta^* = Ad_g(\beta)$$

$$\theta(g^{-1}) = -Ad_g^* \theta(g)$$

$$\Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda$$

- The set Ω of Geometric Temperature is invariant by the adjoint action of G

$$\Psi_g(\mu_\beta) = \mu_{Ad_g(\beta)}$$

- If we use $Q = \dfrac{\partial \Phi}{\partial \beta}$, we have the constraint $\delta\Phi - \langle Q, \delta\beta \rangle = 0$

- By derivation of (**), we have: $\tilde{\Theta}(\beta, Z) + \langle Q, [\beta, Z] \rangle = 0$

$$\begin{aligned} \tilde{\Theta}(X, Y) : \mathfrak{g} \times \mathfrak{g} &\to \Re \\ X, Y &\mapsto \langle \Theta(X), Y \rangle \end{aligned}$$

$$\Theta(X) = T_e \theta(X(e))$$

Geometric (Planck) Temperature

- We have previously observed that: $\tilde{\Theta}(\beta,Z)+\left\langle Q,[\beta,Z]\right\rangle=0$

- $\tilde{\Theta}(X,Y)$ is called the **Symplectic Cocycle of Lie algebra** $\mathfrak{g}$ associated to the momentum map J

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_X,J_Y\} \text{ with } \{.,.\} \text{Poisson Bracket and } J \text{ the Moment Map}$$

$$\mathfrak{g}\to C^{\infty}(M,R)$$
$$X\to J_X$$

> where J_X linear application from $\mathfrak{g}$ to differential function on M:
>
> and the associated differentiable application J, called moment(um) map:

$$J:M\to\mathfrak{g}^* \text{ with } x\mapsto J(x) \text{ such that } J_X(x)=\left\langle J(x),X\right\rangle,\ X\in\mathfrak{g}$$

- $\tilde{\Theta}(X,Y)$ is a 2-form of $\mathfrak{g}$ and verify:

$$\tilde{\Theta}([X,Y],Z)+\tilde{\Theta}([Y,Z],X)+\tilde{\Theta}([Z,X],Y)=0$$

- If we define: $\tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\left\langle Q,ad_{Z_1}(Z_2)\right\rangle$ with $ad_{Z_1}(Z_2)=[Z_1,Z_2]$

- We can observe that : $\beta\in Ker\,\tilde{\Theta}_\beta$ $\quad \tilde{\Theta}_\beta(\beta,\beta)=0 \ ,\ \forall\beta\in\mathfrak{g}$

Associated Riemannian Metric: Geometric Fisher Metric

- We can compute the image of Geometric Heat by the Lie Group action:

$$Q^* = Ad_g^*(Q) + \theta(g)$$

- By tangential derivative to the orbit with respect to $Z \in \mathfrak{g}$ and by using positivity of $-\dfrac{\partial Q}{\partial \beta} \geq 0$, we find:

$$\tilde{\Theta}_\beta(Z,[\beta,Z]) = \tilde{\Theta}(Z,[\beta,Z]) + \langle Q,[Z,[\beta,Z]]\rangle \geq 0$$

- $\tilde{\Theta}_\beta$ is a 2-form of $\mathfrak{g}$ that verifies:

$$\tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([Y,Z],X) + \tilde{\Theta}([Z,X],Y) = 0$$

- Then, there exists a **symmetric tensor** g_β defined on $ad_\beta(Z)$

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

- With the following invariances:

$$I(Ad_g(\beta)) = -\frac{\partial^2\left(\Phi - \langle \theta(g^{-1}),\beta\rangle\right)}{\partial\beta^2} = -\frac{\partial^2\Phi}{\partial\beta^2} = I(\beta)$$

$$s[Q(Ad_g(\beta))] = s(Q(\beta))$$

Fisher Metric of Souriau Lie Group Thermodynamics

- Souriau has introduced the Riemannian metric

$$g_\beta([\beta, Z_1], [\beta, Z_2]) = \tilde{\Theta}_\beta(Z_1, [\beta, Z_2]) \qquad \beta \in Ker\, \tilde{\Theta}_\beta$$

$$\tilde{\Theta}_\beta(Z_1, Z_2) = \tilde{\Theta}(Z_1, Z_2) + \left\langle Q, ad_{Z_1}(Z_2) \right\rangle \text{ with } ad_{Z_1}(Z_2) = [Z_1, Z_2]$$

- This metric is an **extension of Fisher metric, an hessian metric**: If we differentiate the relation $Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g)$

$$\frac{\partial Q}{\partial \beta}(-[Z_1, \beta], .) = \tilde{\Theta}(Z_1, [\beta, .]) + \left\langle Q, Ad_{.Z_1}([\beta, .]) \right\rangle = \tilde{\Theta}_\beta(Z_1, [\beta, .])$$

$$-\frac{\partial Q}{\partial \beta}([Z_1, \beta], Z_2.) = \tilde{\Theta}(Z_1, [\beta, Z_2]) + \left\langle Q, Ad_{.Z_1}([\beta, Z_2]) \right\rangle = \tilde{\Theta}_\beta(Z_1, [\beta, Z_2])$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial \beta^2} = -\frac{\partial Q}{\partial \beta} = g_\beta([\beta, Z_1], [\beta, Z_2]) = \tilde{\Theta}_\beta(Z_1, [\beta, Z_2])$$

- The Fisher Metric is then a **generalization of "Heat Capacity"**:

$$\beta = \frac{1}{kT} \qquad K = -\frac{\partial Q}{\partial \beta} = -\frac{\partial Q}{\partial T}\left(\frac{\partial (1/kT)}{\partial T}\right)^{-1} = kT^2 \frac{\partial Q}{\partial T} \qquad \frac{\partial T}{\partial t} = \frac{\kappa}{C.D}\Delta T \text{ with } \frac{\partial Q}{\partial T} = C.D$$

Continuous Medium Thermodynamics

- For Continuous Medium Thermodynamics, « Temperature Vector » is no longer constrained to be in Lie Algebra, but only contrained by phenomenologic equations (e.g. Navier equations, …).

- For Thermodynamic equilibrium, the **« Temperature Vector » is a Killing vector of Space-Time**.

- For each point X, there is a « Temperature Vector » $\beta(X)$, such it is an infinitesimal conformal transform of the metric of the univers g_{ij} :

$$\hat{\partial}_i \beta_j + \hat{\partial}_j \beta_i = \lambda g_{ij}$$
$$\partial_i \beta_j + \partial_j \beta_i - 2\Gamma_{ij}^k \beta_k = \lambda g_{ij}$$
$$\lambda = 0 \Rightarrow \text{Killing Equation}$$

$$\begin{cases} \hat{\partial}_i . : \text{covariant derivative} \\ \beta_j : \text{component of Temperature vector} \end{cases}$$

- Conservation equations can be deduced for components of Impulsion-Energy tensor T^{ij} and Entropy flux S^j : $\hat{\partial}_i T^{ij} = 0 \quad \partial_i S^j = 0$

Poincaré-Cartan Integral Invariant of Lie Group Thermodynamics

- Analogies between Geometric Mechanics & Geometric Lie Group Thermodynamics, provides the following similarities of structures:

$$\begin{cases} \dot{q} \leftrightarrow \beta \\ p \leftrightarrow Q \end{cases} \qquad \begin{cases} L(\dot{q}) \leftrightarrow \Phi(\beta) \\ H(p) \leftrightarrow S(Q) \\ H = p.\dot{q} - L \leftrightarrow S = \langle Q, \beta \rangle - \Phi \end{cases} \qquad \begin{cases} \dot{q} = \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \leftrightarrow \beta = \dfrac{\partial S}{\partial Q} \\ p = \dfrac{\partial L}{\partial \dot{q}} \leftrightarrow Q = \dfrac{\partial \Phi}{\partial \beta} \end{cases}$$

- We can then consider a similar Poincaré-Cartan-Souriau Pfaffian form:

$$\omega = p.dq - H.dt \leftrightarrow \omega = \langle Q, (\beta.dt) \rangle - S.dt = \left(\langle Q, \beta \rangle - S \right).dt = \Phi(\beta).dt$$

- This analogy provides an associated Poincaré-Cartan Integral Invariant:

$$\int_{C_a} p.dq - H.dt = \int_{C_b} p.dq - H.dt \quad \text{transforms in} \quad \int_{C_a} \Phi(\beta).dt = \int_{C_b} \Phi(\beta).dt$$

- For Thermodynamics, we can then deduce an Euler-Poincaré Variational Principle: The Variational Principle holds on $\mathfrak{g}$, for variations $\delta\beta = \dot{\eta} + [\beta, \eta]$, where $\eta(t)$ is an arbitrary path that vanishes at the endpoints, $\eta(a) = \eta(b) = 0$:

$$\delta \int_{t_0}^{t_1} \Phi(\beta(t)).dt = 0$$

Souriau Gibbs states for Hamiltonian actions of subgroups of the Galilean group

- Galilean Transformation on position and speed:

$$\begin{pmatrix} \vec{r}\,' & \vec{v}\,' \\ t' & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & \vec{b} & \vec{d} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{r} & \vec{v} \\ t & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A\vec{r} + t\vec{b} + \vec{d} & A\vec{v} + \vec{b} \\ t + e & 1 \\ 1 & 0 \end{pmatrix}$$

- **<u>Souriau Result</u>**: this action is Hamiltonian, with the map ***J***, defined on the evolution space of the particle, with value in the dual $\mathfrak{g}^*$ of the Lie algebra ***G***, as momentum map

$$J(\vec{r},t,\vec{v},m) = m \begin{pmatrix} \vec{r}\times\vec{v} & 0 & 0 \\ \vec{r}-t\vec{v} & 0 & 0 \\ \vec{v} & \frac{1}{2}\|\vec{v}\|^2 & 0 \end{pmatrix} = m\left\{\vec{r}\times\vec{v}, \vec{r}-t\vec{v}, \vec{v}, \frac{1}{2}\|\vec{v}\|^2\right\} \in \mathfrak{g}^*$$

- Couplling formula:

$$\langle J(\vec{r},t,\vec{v},m), \beta \rangle = \left\langle m\left\{\vec{r}\times\vec{v}, \vec{r}-t\vec{v}, \vec{v}, \frac{1}{2}\|\vec{v}\|^2\right\}, \left\{\vec{\omega}, \vec{\alpha}, \vec{\delta}, \varepsilon\right\} \right\rangle$$

$$Z = \begin{pmatrix} j(\vec{\omega}) & \vec{\alpha} & \vec{\delta} \\ 0 & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} = \left\{\vec{\omega}, \vec{\alpha}, \vec{\delta}, \varepsilon\right\} \in \mathfrak{g}$$

$$\langle J(\vec{r},t,\vec{v},m), \beta \rangle = m\left(\vec{\omega}.\vec{r}\times\vec{v} - (\vec{r}\times\vec{v}).\vec{\alpha} + \vec{v}.\vec{\delta} - \frac{1}{2}\|\vec{v}\|^2 \varepsilon \right)$$

Souriau Gibbs states for one-parameter subgroups of the Galilean group

- **Souriau Result:** Action of the full Galilean group on the space of motions of an isolated mechanical system is not related to any Equilibrium Gibbs state (the open subset of the Lie algebra, associated to this Gibbs state, is empty)
- The **1-parameter subgroup of the Galilean group** generated by β element of Lie Algebra, is the set of matrices

$$\exp(\tau\beta)=\begin{pmatrix} A(\tau) & \vec{b}(\tau) & \vec{d}(\tau) \\ 0 & 1 & \tau\varepsilon \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} A(\tau)=\exp(\tau j(\vec{\omega})) \text{ and } \vec{b}(\tau)=\left(\sum_{i=1}^{\infty}\frac{\tau^i}{i!}(j(\vec{\omega}))^{i-1}\right)\vec{\alpha} \\ \vec{d}(\tau)=\left(\sum_{i=1}^{\infty}\frac{\tau^i}{i!}(j(\vec{\omega}))^{i-1}\right)\vec{\delta}+\varepsilon\left(\sum_{i=2}^{\infty}\frac{\tau^i}{i!}(j(\vec{\omega}))^{i-2}\right)\vec{\alpha} \end{cases}$$

$$\beta=\begin{pmatrix} j(\vec{\omega}) & \vec{\alpha} & \vec{\delta} \\ 0 & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}\in\mathfrak{g}$$

Souriau Thermodynamics of butter churn (device used to convert cream into butter) or “La Thermodynamique de la crémière”

If we consider the case of the centrifuge

$$\vec{\omega} = \omega \vec{e}_z \ , \vec{\alpha} = 0 \ \text{ and } \ \vec{\delta} = 0$$

$$Rotation\ speed: \frac{\omega}{\varepsilon}$$

$$f_i\left(\vec{r}_{i0}\right) = -\frac{\omega^2}{2\varepsilon^2}\left\|\vec{e}_z \times \vec{r}_{i0}\right\|^2$$

$$\text{with } \ \Delta = \left\|\vec{e}_z \times \vec{r}_{i0}\right\| \ \text{ distance to axis z}$$

> “The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”. – Roger Balian

$$\rho_i(\beta) = \frac{1}{P_i(\beta)}\exp\left(-\left\langle J_i, \beta \right\rangle\right) = cst.\exp\left(-\frac{1}{2m_i\kappa T}\left\|\vec{p}_{i0}\right\|^2 + \frac{m_i}{2\kappa T}\left(\frac{\omega}{\varepsilon}\right)^2 \Delta^2\right)$$

- the behaviour of a gas made of point particles of various masses in a centrifuge rotating at a constant angular velocity (the heavier particles concentrate farther from the rotation axis than the lighter ones)

$$\frac{\omega}{\varepsilon}$$

Roger Balian Computation of Gibbs density for centrifuge

- Balian made the remarks that ***“The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”.***

Lie Group Thermodynamics: Centrifuge for Butter, U235 & Ribo acides

- Physiquement, la théorie donne de bons résultats si on l'applique aux divers sous-groupes du groupe de Galilée qui sont caractéristiques des appareils thermodynamiques : ainsi une boite cylindrique dans laquelle on enferme un fluide lui laisse un sous-groupe d'invariance de dimension 2 : rotations autour de l'axe, translations temporelles. D'où résulte un vecteur température à deux dimensions, que l'on peut "transmettre" au fluide par l'intermédiaire de la boite, (en la refroidissant, par exemple, et en la faisant tourner) ; les résultats de la théorie sont ceux-là même que l'on exploite dans les centrifugeuses (par exemple pour fabriquer du beurre, de l'uranium 235 ou des acides ribonucléiques).

- On remarquera que le processus par lequel une centrifugeuse réfrigérée transmet son propre vecteur-température à son contenu porte deux noms différents : conduction thermique et viscosité, selon la composante du vecteur-température que l'on considère ; conduction et viscosité devraient donc être unifiées dans une théorie fondamentale des processus irréversibles (théorie qui reste à construire).

THALES

OPEN

THALES

Gibbs Density on Poincaré Unit Disk from Souriau Lie Groups Thermodynamics and SU(1,1) Coadjoint Orbits

Poincaré Unit Disk and SU(1,1) Lie Group

- The group of complex unimodular pseudo-unitary matrices $SU(1,1)$:

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1,\ a, b \in C \right\}$$

- the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$\mathfrak{g} = \left\{ \begin{pmatrix} -ir & \eta \\ \eta^* & ir \end{pmatrix} / r \in R, \eta \in C \right\}$$

with the following bases $(u_1, u_2, u_3) \in \mathfrak{g}$:

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

with the commutation relation:

$$[u_3, u_2] = u_1, [u_3, u_1] = u_2, [u_2, u_1] = -u_3$$

Poincaré Unit Disk and SU(1,1) Lie Group

- Dual base on dual Lie algebra is named
$$\left(u_1^*, u_2^*, u_3^*\right) \in \mathfrak{g}^*$$
- The dual vector space $\mathfrak{g}^* = \mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2,C)$ of the form:

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / \, x, y, z \in R \right\}$$

- Coadjoint action of $g \in G$ on dual Lie algebra $\xi \in \mathfrak{g}^*$ is written $g.\xi$

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

- The torus $K=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in R\right\}$ induces rotations of the unit disk
- K leaves 0 invariant. The stabilizer for the origin 0 of unit disk is maximal compact subgroup K of $SU(1,1)$.
- B. Cahen has observed that $O\left(ru_3^*\right) \approx G/K$ and is diffeomorphic to the unit disk $D=\left\{z \in C / |z| < 1\right\}$
- The **moment map** is given by:

$$J : D \to O\left(ru_3^*\right)$$

$$z \mapsto J(z) = r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)}u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right)$$

Benjamin Cahen, Contraction de SU(1,1) vers le groupe de Heisenberg, Travaux mathématiques, Fascicule XV, pp.19-43, (2004)

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

$$J : D \to O_n$$

$$z \mapsto J(z) = \frac{n}{2}\left(\frac{z+z^*}{\left(1-|z|^2\right)} u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)} u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)} u_3^* \right)$$

- Group G act on D by homography: $g.z = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} .z = \dfrac{az+b}{a^* z + b^*}$
- This action corresponds with coadjoint action of G on O_n .
- The Kirillov-Kostant-Souriau 2-form of O_n is given by:

$$\Omega_n(\zeta)\left(X(\zeta), Y(\zeta)\right) = \left\langle \zeta, [X, Y] \right\rangle \text{ , } X, Y \in \mathfrak{g} \text{ and } \zeta \in O_n$$

- and is associated in the frame by ψ_n with:

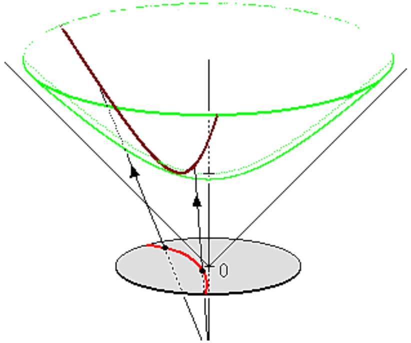
$$\omega_n = \frac{in}{\left(1-|z|^2\right)^2} dz \wedge dz^*$$

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

$$J(z)=r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^*+\frac{z-z^*}{i\left(1-|z|^2\right)}u_2^*+\frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right)\in O\left(ru_3^*\right),z\in D$$

- J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $O\left(ru_3^*\right)=G/K$
- J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $C\cup\infty$:

The coadjoint action of $G=SU(1,1)$ is the upper sheet $x_3>0$ of the two-sheet hyperboloid



Charles-Michel Marle, Projection stéréographique et moments, hal-02157930, version 1, Juin 2019

$$\left\{\xi=x_1u_1^*+x_2u_2^*+x_3u_3^*:-x_1^2-x_2^2+x_3^2=r^2\right\}$$

Covariant Gibbs Density by Souriau Thermodynamics

- We can use Kirillov representation theory and his character formula to compute Souriau covariant Gibbs density in the unit Poincaré disk.
- For any Lie group G, a coadjoint orbit $\mathrm{O} \subset \mathfrak{g}^*$ has a canonical symplectic form given by KKS 2-form ω_{O} .
- If is finite dimensional, the corresponding volume element defines a G-invariant measure supported on O, which can be interpreted as a tempered distribution.
- The Fourier transform : $$\Im(x) = \int_{\mathrm{O}\subset\mathfrak{g}^*} e^{-i\langle x,\lambda\rangle} \frac{1}{d!} d\omega^d \quad \text{with } \lambda \in \mathfrak{g}^* \quad \text{and} \quad x \in \mathfrak{g}$$
- is Ad G-invariant. When $\mathrm{O} \subset \mathfrak{g}^*$ is an integral coadjoint orbit, Kirillov has proved that this Fourier transform is related to Kirillov character χ_{O} by:

$$\Im(x) = j(x)\chi_{\mathrm{O}}\left(e^x\right) \quad \text{where } j(x) = \det{}^{1/2}\left(\frac{\sinh\left(ad\left(x/2\right)\right)}{ad\left(x/2\right)}\right)$$

Symplectic Metric of Unit Disk

Symplectic Homogeneous Manifold

- Let consider $D=\{z\in C/|z|<1\}$ be the open unit disk of Poincaré. For each $\rho>0$, the pair (D,ω_ρ) is a symplectic homogeneous manifold with:

$$\omega_\rho=2i\rho\frac{dz\wedge dz^*}{\left(1-|z|^2\right)^2}$$

where ω_ρ is invariant under the action : $SU(1,1)\times D\rightarrow D$

$$(g,z)\mapsto g.z=\frac{az+b}{b^*z+a^*}$$

- This action is transitive and is globally and strongly Hamiltonian. Its generators are the hamiltonian vector fields associated to the functions:

$$J_1(z,z^*)=\rho\frac{1+|z|^2}{1-|z|^2},J_2(z,z^*)=\frac{\rho}{i}\frac{z-z^*}{1-|z|^2},\ J_3(z,z^*)=-\rho\frac{z+z^*}{1-|z|^2}$$

Moment Map for SU(1,1)

Invariant Moment Map

- The associated moment map $J : D \to su^*(1,1)$ defined by $J(z).u_i = J_i(z,z^*)$, maps D into a coadjoint orbit in $su^*(1,1)$.
- Then, we can write the moment map as a matrix element of $su^*(1,1)$:

$$J(z) = J_1\left(z,z^*\right)u_1^* + J_2\left(z,z^*\right)u_2^* + J_3\left(z,z^*\right)u_3^*$$

$$J(z) = \rho \begin{pmatrix} \dfrac{1+|z|^2}{1-|z|^2} & -2\dfrac{z^*}{1-|z|^2} \\ 2\dfrac{z}{1-|z|^2} & -\dfrac{1+|z|^2}{1-|z|^2} \end{pmatrix} \in \mathfrak{g}^*$$

Moment Map & Stereographic projection

One sheet of the two-sheeted hyperboloid

- The moment map J is a diffeomorphism of D onto one sheet of the two-sheeted hyperboloid in $su^*(1,1)$, determined by the following equation:

$$J_1^2 - J_2^2 - J_3^2 = \rho^2 \quad , \; J_1 \geq \rho \;\; \text{with} \quad J_1 u_1^* + J_2 u_2^* + J_3 u_3^* \in su^*(1,1)$$

- We note O_ρ^+ the coadjoint orbit $Ad^*_{SU(1,1)}$ of $SU(1,1)$, given by the upper sheet of the two-sheeted hyperboloid given by previous equation.
- The orbit method of Kostant-Kirillov-Souriau associates to each of these coadjoint orbits a representation of the discrete series of $SU(1,1)$, provided that ρ is a half integer greater or equal than 1: $\rho = k/2, k \in N$ and $\rho \geq 1$
- When explicitly executing the Kostant-Kirillov construction, the representation Hilbert spaces H_ρ are realized as closed reproducing kernel subspaces of $L^2(D, \omega_\rho)$ .The Kostant-Kirillov-Souriau orbit method shows that to each coadjoint orbit of a connected Lie group is associated a unitary irreducible representation of G acting in a Hilbert space H.

Gibbs state Equilibrium

One parameter subgroup

- Souriau has oberved that action of the full Galilean group on the space of motions of an isolated mechanical system is not related to any equilibrium Gibbs state (the open subset of the Lie algebra, associated to this Gibbs state is empty).
- The main Souriau idea was to define the Gibbs states for one-parameter subgroups of the Galilean group. We will use the same approach, in this case We will consider action of the Lie group $SU(1,1)$ on the symplectic manifold (M,ω) (Poincaré unit disk) and its momentum map J are such that the following open subset is not empty:

$$\Lambda_\beta = \left\{ \beta \in \mathfrak{g} / \int_D e^{-\langle J(z),\beta \rangle} d\lambda(z) < +\infty \right\}$$

- The idea of Souriau is to consider a one parameter subgroup of $SU(1,1)$. To parametrize elements of $SU(1,1)$ is through its Lie algebra. In the neighborhood of the identity element, the elements of $g \in SU(1,1)$ can be written as the exponential of an element β of its Lie algebra : $g = \exp(\varepsilon\beta)$ with $\beta \in \mathfrak{g}$

Gibbs State Equilibrium

One parameter subgroup

- We can then exponentiate β with exponential map to get :

$$g=\exp(\varepsilon\beta)=\sum_{k=0}^{\infty}\frac{(\varepsilon\beta)^k}{k!}=\begin{pmatrix} a_\varepsilon(\beta) & b_\varepsilon(\beta) \\ b_\varepsilon^*(\beta) & a_\varepsilon^*(\beta) \end{pmatrix}$$

- If we make the remark that we have the following relation

$$\beta^2=\begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}\begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}=\left(|\eta|^2-r^2\right)I$$

- we can developed the exponential map :

$$g=\exp(\varepsilon\beta)=\begin{pmatrix} \cosh(\varepsilon R)+ir\dfrac{\sinh(\varepsilon R)}{R} & \eta\dfrac{\sinh(\varepsilon R)}{R} \\ \eta^*\dfrac{\sinh(\varepsilon R)}{R} & \cosh(\varepsilon R)-ir\dfrac{\sinh(\varepsilon R)}{R} \end{pmatrix} \text{ with } R^2=|\eta|^2-r^2$$

Gibbs State Equilibrium

Equilibrium conditions

- We can observe that one condition is that $|\eta|^2 - r^2 > 0$
- Then the subset to consider is given by the subset

$$\Lambda_\beta = \left\{ \beta = \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}, r \in R, \eta \in C \,/\, |\eta|^2 - r^2 > 0 \right\}$$

- such that: $\int_D e^{-\langle J(z),\beta \rangle} d\lambda(z) < +\infty$
- The generalized Gibbs states of the full $SU(1,1)$ group do not exist. However, generalized Gibbs states for the one-parameter subgroups $\exp(\alpha\beta)$, $\beta \in \Lambda_\beta$ of the $SU(1,1)$ group do exist.
- The generalized Gibbs state associated to β remains invariant under the restriction of the action to the one-parameter subgroup of $SU(1,1)$ generated by $\exp(\varepsilon\beta)$.

Souriau Gibbs density for SU(1,1)

Covariant Gibbs density

- We can the write the covariant Gibbs density in the unit disk given by moment map of the Lie group $SU(1,1)$ and geometric temperature in its Lie algebra $\beta \in \Lambda_\beta$:

$$p_{Gibbs}(z) = \frac{e^{-\langle J(z),\beta \rangle}}{\int_D e^{-\langle J(z),\beta \rangle} d\lambda(z)} \text{ with } d\lambda(z) = 2i\rho \frac{dz \wedge dz^*}{\left(1-|z|^2\right)^2}$$

$$p_{Gibbs}(z) = \frac{e^{-\left\langle \rho\left(2\Im bb^+ - Tr\left(\Im bb^+\right)I\right),\beta\right\rangle}}{\int_D e^{-\langle J(z),\beta \rangle} d\lambda(z)} = \frac{e^{-\left\langle \rho \begin{pmatrix} \frac{1+|z|^2}{\left(1-|z|^2\right)} & \frac{-2z^*}{\left(1-|z|^2\right)} \\ \frac{2z}{\left(1-|z|^2\right)} & -\frac{1+|z|^2}{\left(1-|z|^2\right)} \end{pmatrix}, \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix} \right\rangle}}{\int_D e^{-\langle J(z),\beta \rangle} d\lambda(z)}$$

$$J(z) == \rho\left(2Mbb^+ - Tr\left(Mbb^+\right)I\right) \text{ with } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } b = \frac{1}{1-|z|^2}\begin{bmatrix} 1 \\ -z \end{bmatrix}$$

Souriau Gibbs density

Covariant Gibbs Density

$$p_{Gibbs}(z)=\frac{e^{-\left\langle \rho\begin{pmatrix}\frac{1+|z|^2}{(1-|z|^2)} & \frac{-2z^*}{(1-|z|^2)}\\ \frac{2z}{(1-|z|^2)} & -\frac{1+|z|^2}{(1-|z|^2)}\end{pmatrix},\begin{pmatrix} ir & \eta\\ \eta^* & -ir\end{pmatrix}\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle}d\lambda(z)}$$

- To write the Gibbs density with respect to its statistical moments, we have to express the density with respect to $Q=E\left[J(z)\right]$
- Then, we have to invert the relation between Q and β, to replace $\beta=\begin{pmatrix} ir & \eta\\ \eta^* & -ir\end{pmatrix}\in\Lambda_\beta$ by $\beta=\Theta^{-1}(Q)\in\mathfrak{g}$ where $Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\Theta(\beta)\in\mathfrak{g}^*$ with $\Phi(\beta)=-\log\int_D e^{-\langle J(z),\beta\rangle}d\lambda(z)$ deduce from Legendre tranform. The mean moment map is given by:

$$Q=E\left[J(z)\right]=E\left[\rho\begin{pmatrix}\frac{1+|w|^2}{(1-|w|^2)} & \frac{-2w^*}{(1-|w|^2)}\\ \frac{2w}{(1-|w|^2)} & -\frac{1+|w|^2}{(1-|w|^2)}\end{pmatrix}\right]\quad \text{where } w\in D$$

THALES

Representation Theory & Orbits Methods: Fourier Transform for Non-Commutative Harmonic analysis

Fourier/Laplace Transform and Representation Theory

- Fourier analysis, named after Joseph Fourier, who showed that representing a function as a sum of trigonometric functions greatly simplifies the study of heat transfer and addresses classically **commutative harmonic analysis**. Classical commutative harmonic analysis is restricted to functions defined on a **topological locally compact and Abelian group G** (Fourier series when **$G = R^n/Z^n$**, Fourier transform when **$G = R^n$**, discrete Fourier transform when G is a finite Abelian group).
- The modern development of Fourier analysis during XXth century has explored the **generalization of Fourier and Fourier-Plancherel formula for non-commutative harmonic analysis**, applied to locally compact **non-Abelian groups**.
- This has been solved by geometric approaches based on "**orbits methods**" (Fourier-Plancherel formula for G is given by coadjoint representation of G in dual vector space of its Lie algebra) with many contributors (Dixmier, Kirillov, Bernat, Arnold, Berezin, Kostant, Souriau, Duflo, Guichardet, Torasso, Vergne, Paradan, etc.)

Dixmier/Kirillov/Duflo/Vergne Representation Theory

Classical Commutative Harmonic Analysis

$G = \mathrm{T}^n = R^n / Z^n$: Fourier series, $\quad G = R^n$: Fourier Transform

G group character (linked to e^{ikx}) : $\begin{cases} \chi : G \to U \\ U = \{ z \in C / |z| = 1 \} \end{cases}$

$$\hat{G} = \{ \chi / \chi_1 . \chi_2 (g) = \chi_1(g) \chi_2(g) \}$$

Fourier Transform

$\varphi : G \to C$

$\hat{\varphi} : \hat{G} \to C$

$$g \mapsto \varphi(g) = \int_{\hat{G}} \hat{\varphi}(\chi) \chi(g)^{-1} d\chi$$

$$\chi \mapsto \hat{\varphi}(\chi) = \int_G \varphi(g) \chi(g) dg$$

Fourier-Plancherel formula

$$\varphi(e) = \int_{\hat{G}} \hat{\varphi}(\chi) d\chi$$

Dixmier/Kirillov/Duflo/Vergne Representation Theory

Character-Distribution

- (Schwarz) Distribution on G : $\chi_{\mathrm{U}}(g) = tr\mathrm{U}_{\varphi}$ with $\mathrm{U}_{\varphi} = \int_G \varphi(g)\mathrm{U}_g dg$

Character Formula: Fourier transform on Lie algebra via Exponential map

$$\mathrm{U}_{\psi} = \int_{\mathfrak{g}} \psi(X)\mathrm{U}_{\exp(X)} dX$$

Kirillov Character :

$$\chi_U\left(\exp(X)\right) = \mathrm{trU}_{\exp(X)} = j(X)^{-1} \int_{\mathrm{O}} e^{i\langle f,X\rangle} d\mu_{\mathrm{O}}(f)$$

Fourier Transform:

$$\int_{\mathrm{O}} e^{i\langle f,X\rangle} d\mu_{\mathrm{O}}(f) = j(X)\mathrm{trU}_{\exp(X)}$$

$$j(X) = \left(\det\left(\frac{e^{ad_X/2} - e^{-ad_X/2}}{ad_X/2}\right)\right)^{1/2}$$

Covariant Gibbs Density by Souriau Thermodynamics

- The moment map is equivariant isomorphism(O_m^+ coadjoint orbit for $m^2>0, E>0$)

$$J: w \in \left(D, curv\left(\kappa^{m/2}\right)\right) \mapsto \left(p, E\right) = \frac{m}{\left(1-|w|^2\right)}\left(2iw, 1+|w|^2\right) \in O_m^+$$

- In case $m>1$, the Kirillov character formula is given by:

$$\chi_m\left(\exp\left(\left(\begin{array}{cc} x & . \\ . & -x \end{array}\right)\right)\right) = j(x)^{-1}\int_{O_{m-1}^+} e^{-i\left\langle\left(\begin{array}{cc} x & . \\ . & -x \end{array}\right),\left(\begin{array}{cc} iE & p^* \\ p & -iE \end{array}\right)\right\rangle}\omega_{O_{m-1}^+}$$

where

$$j(x) = \det{}^{1/2}\left[\sinh\left(ad\left(\begin{array}{cc} x/2 & \\ & -x/2 \end{array}\right)\right) / ad\left(\begin{array}{cc} x/2 & \\ & -x/2 \end{array}\right)\right] = \frac{\sinh(x)}{x}$$

which reduces to :

$$\frac{e^{mx}}{1-e^{2x}}j(x) = \int_D e^{(m-1)x\frac{1+|w|^2}{1-|w|^2}}\frac{1}{\left(1-|w|^2\right)^2}dw \wedge dw^*$$

THALES

EXTENSION: GIBBS DENSITY FOR SU(n,n) in Siegel Disk

Symplectic Group(Carl-Ludwig Siegel) : Siegel Upper half space SH_n

Siegel metric on the Siegel Upper Half Space:

- Siegel Upper half Space: $\boxed{SH_n = \{Z = X + iY \in Sym(n,C) / \operatorname{Im}(Z) = Y > 0\}}$
- Isometries of SH_n are given by quotient Group:

$PSp(n,R) \equiv Sp(n,R)/\{\pm I_{2n}\}$ with $Sp(n,F)$ Symplectic Group:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow M(Z) = (AZ+B)(CZ+D)^{-1}$$

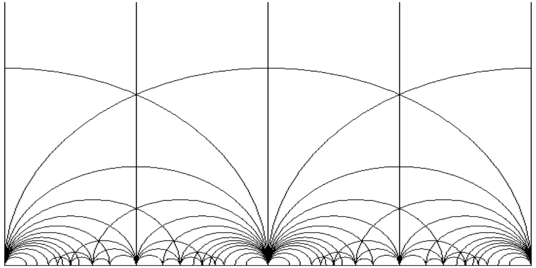
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,F) \Leftrightarrow \begin{cases} A^T C \text{ et } B^T D \quad \text{symmetric} \\ A^T D - C^T B = I_n \end{cases}$$

$$Sp(n,F) \equiv \{M \in GL(2n,F) / M^T J M = J\},\ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in SL(2n,R)$$

- Metric invariant by the automorpisms $M(Z)$:

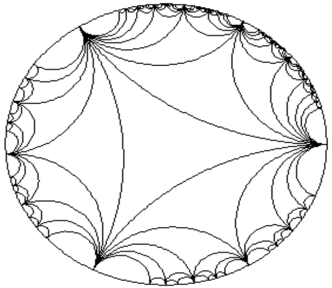
$$ds^2_{Siegel} = Tr\left(Y^{-1}(dZ)Y^{-1}(d\bar{Z})\right) \qquad Z = X + iY$$

Extension of homogeneous bounded symmetric domains: Siegel Upper half-space and Siegel disk



Poincaré Upper half Plane

$$w = \frac{z-i}{z+i}$$



Poincaré Unit Disk

$$ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{|dz|^2}{y^2}$$

$$ds^2 = y^{-1}dzy^{-1}dz^*$$

with $z = x+iy$ and $y>0$

$$ds^2 = \frac{|dw|^2}{\left(1-|w|^2\right)^2}$$

$$ds^2 = \left(1-ww^*\right)^{-1}dw\left(1-ww^*\right)^{-1}dw^*$$

Siegel Upper Half Space

$$W = (Z-iI)(Z+iI)^{-1}$$

Siegel Unit Disk

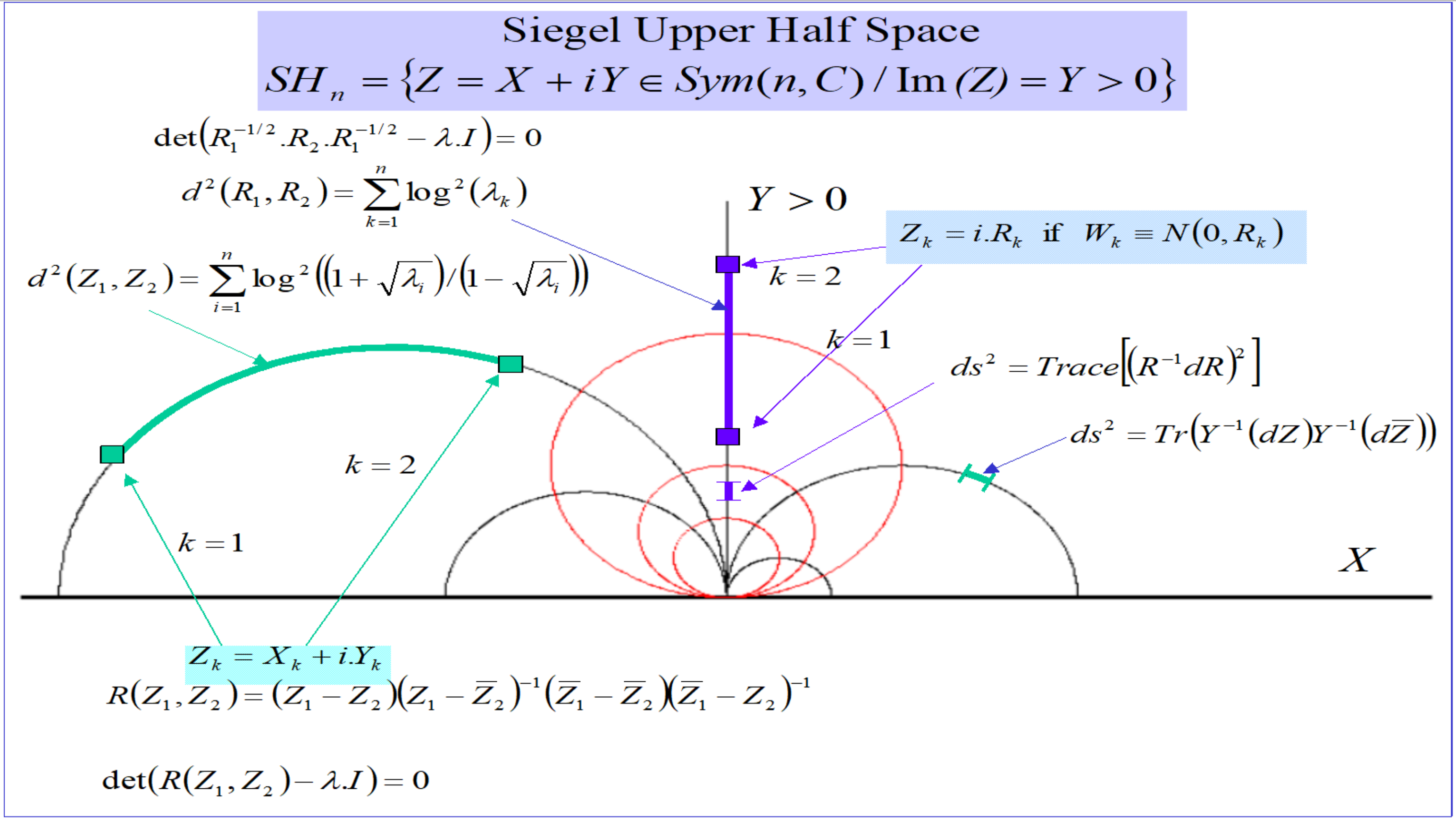
$$ds^2 = Tr\left(Y^{-1}dZY^{-1}d\bar{Z}\right)$$

with $Z = X+iY$

$X \in Herm(n,C)$ and $Y \in HPD(n,C)$

$$ds^2 = Tr\left[\left(I-WW^+\right)^{-1}dW\left(I-W^+W\right)^{-1}dW^+\right]$$

Extension of homogeneous Bounded symmetric domains: Siegel Upper half-space and Siegel disk



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Cartan-Siegel Symmetric Homogeneous Bounded Domains

$$\Omega^{I}_{1,1} = \Omega^{II}_{1} = \Omega^{III}_{1} = \Omega^{IV}_{1} = \left\{ z \in C / zz^* < 1 \right\}, K(z,w^*) = \frac{1}{(1 - zw^*)^2}$$

Z : Complex Rectangular Matrix

$\boxed{ZZ^+ < I}$ $({}^+ : transposed - conjugate)$

Type I : $\Omega^{I}_{p,q}$ complex matrices with p lines and q rows

$\boxed{\text{Type II}}$: Ω^{II}_{p} complex symmetric matrices of order p

Type III : Ω^{III}_{p} complex skew symmetric matrices of order p

Type IV : Ω^{IV}_{n} complex matrices with n rows and 1 line :

$$\left|ZZ^t\right| < 1, 1 + \left|ZZ^t\right|^2 - 2ZZ^+ > 0$$

$$K(Z,W^*) = \frac{1}{\mu(\Omega)} \det(I - ZW^+)^{-\nu} \quad \text{for} \quad \begin{cases} \text{Type I} : \Omega^{I}_{p,q}, \nu = p + q \\ \text{Type II} : \Omega^{II}_{p}, \nu = p + 1 \\ \text{Type III} : \Omega^{III}_{p}, \nu = p - 1 \end{cases}$$

$$K(Z,W^*) = \frac{1}{\mu(\Omega)} \left(1 + ZZ^t W^* W^+ - 2ZW^*\right)^{-\nu} \text{ for Type IV} : \Omega^{IV}_{n}, \nu = n$$

where $\mu(\Omega)$ is euclidean volume of the domain.

Henri Poincaré (n=1)

Elie Cartan (n<=3)

Carl Ludwig Siegel

Lookeng Hua

THALES

Extension for Gibbs density on Siegel Unit Disk

From Poincaré Unit Disk to Siegel Unit Disk

- To extend this approach for covariant Gibbs density on Siegel Unit Disk:

$$SD = \left\{ Z \in M_{pq}\left(C\right) / I_p - ZZ^+ > 0 \right\}$$

- that is a classical matrix extension of Poincaré unit Disk, we have proposed to consider $G = SU(p,q)$ unitary group and the homogeneous space :

$$G / K = SU(p,q) / S\left(U(p),U(q)\right)$$

$$\text{with } K = S\left(U(p) \times U(q)\right) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} / A \in U(p), D \in U(q), \det(A)\det(D) = 1 \right\}$$

Extension for Gibbs density on Siegel Unit Disk

- We can use the following decomposition for $g \in G^C$

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G^C, g = \begin{pmatrix} I_p & BD^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_p & 0 \\ D^{-1}C & I_q \end{pmatrix}$$

- and consider the action of $g \in G^C$ on Siegel Unit Disk given by:

$$SD = \left\{ Z \in M_{pq}(C) / I_p - ZZ^+ > 0 \right\}$$

Les Houches 27th-31st July 2020
Joint Structures and Common Foundations of Statistical Physics,
Information Geometry and Inference for Learning (SPIGL'20)

THALES

Extension for Gibbs density on Siegel Unit Disk

- Benjamin Cahen has study this case and introduced the moment map by identifing G-equivariantly $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the Killing form β on $\mathfrak{g}^C$:

$$\mathfrak{g}^* \text{ G-equivariant with } \mathfrak{g} \text{ by Killing form } \beta(X,Y)=2(p+q)Tr(XY)$$

- The set of all elements of $\mathfrak{g}$ fixed by K is $\mathfrak{h}$:

$$\mathfrak{h}=\{\text{element of } G \text{ fixed by } K\} \quad , \xi_0 \in \mathfrak{h}, \xi_0 = i\lambda\begin{pmatrix} -qI_p & 0 \\ 0 & pI_q \end{pmatrix}$$

$$\Rightarrow \left\langle \xi_0, \left[Z, Z^+\right]\right\rangle = -2i\lambda(p+q)^2 Tr\left(ZZ^+\right), \forall Z \in D$$

- Then, we the equivatiant moment map is given by:

$$\forall X \in g^C \ , Z \in D, \quad \psi(Z) = Ad^*\left(\exp\left(-Z^+\right)\zeta\left(\exp Z^+ \exp Z\right)\right)\xi_0$$

$$\zeta\left(\exp Z^+ \exp Z\right) = \begin{pmatrix} I_p & Z\left(I_q - Z^+Z\right)^{-1} \\ 0 & I_q \end{pmatrix}. \ \forall g \in G, Z \in D \text{ then}$$

$$\psi(g.Z) = Ad_g^*\psi(Z): \ \psi \text{ is a diffeomorphism from } SD \text{ onto orbit } O(\xi_0)$$

Extension for Gibbs density on Siegel Unit Disk

From Poincaré Unit Disk to Siegel Unit Disk

- To extend this approach for covariant Gibbs density on Siegel Unit Disk:

$$SD = \left\{ Z \in M_{pq}(C) / I_p - ZZ^+ > 0 \right\}$$

- that is a classical matrix extension of Poincaré unit Disk, we have proposed to consider $G = SU(p,q)$ unitary group and the homogeneous space :

$$G/K = SU(p,q)/S\left(U(p),U(q)\right)$$

$$\text{with } K = S\left(U(p)\times U(q)\right) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} / A \in U(p), D \in U(q), \det(A)\det(D) = 1 \right\}$$

- The moment map given by:

$$J(Z) = i\lambda \begin{pmatrix} \left(I_p - ZZ^+\right)^{-1}\left(-pZZ^+ - qI_p\right) & (p+q)Z\left(I_q - Z^+Z\right)^{-1} \\ -(p+q)\left(I_q - Z^+Z\right)^{-1}Z^+ & \left(pI_q + qZ^+Z\right)\left(I_q - Z^+Z\right)^{-1} \end{pmatrix}$$

THALES

GIBBS DENSITY FOR SE(2) LIE GROUPS FOR MACHINE LEARNING ON KINEMATICS

Coadjoint action of SE(2)

- We will consider Souriau model for $SE(2)$ Lie group with non-null cohomology and then with introduction of Souriau one-cocycle.
- We consider $SE(2)=SO(2)\times R^2$:

$$SE(2)=\left\{\begin{bmatrix} R_\varphi & \tau \\ 0 & 1 \end{bmatrix} / R_\varphi \in SO(2), \tau \in R^2\right\}$$

- The Lie algebra $se(2)$ of $SE(2)$ has underlying vector space R^3 and Lie bracket:

$$(\xi,u)\in se(2)=R\times R^2, \begin{bmatrix} -\xi\Im & u \\ 0 & 0 \end{bmatrix} \in se(2) \text{ with } \Im=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Coadjoint action of $SE(2)$ is given by:

$$Ad^*_{(R_\varphi,\tau)}(m,\rho)=(m+\Im R_\varphi \rho.\tau, R_\varphi \rho)$$

Moment map Computation for SE(2)

- Let $J_{(\xi,u)}(x): R^2 \to se^*(2)$ be the moment map of this action relative to the symplectic form, we can compute it from its definition:

$$dJ_{(\xi,u)}(x).y = -2\omega\left((\xi,u)_{R^2}, y\right)$$

$$\text{with} \quad \omega\left((\xi,u)_{R^2}, y\right) = \omega(\xi\Im x - u, y) = (\xi\Im x - u).\Im y = (\xi x + \Im u).y$$

$$\Rightarrow dJ_{(\xi,u)}(x).y = -2(\xi x + \Im u).y$$

$$\Rightarrow J_{(\xi,u)}(x) = -2\left(\frac{1}{2}\xi\|x\|^2 + \Im u.x\right) = -2\left(\frac{1}{2}\|x\|^2, -\Im x\right).(\xi,u)$$

$$J_{(\xi,u)}(x) = J(x).(\xi,u) \Rightarrow J(x) = -2\left(\frac{1}{2}\|x\|^2, -\Im x\right) \ , \ x \in R^2$$

Souriau moment map for SE(2)

- The moment map $J:R^2 \to se^*(2)$ of $SE(2)$ is defined by:
  $J_{(\xi,u)}(x)=J(x).(\xi,u)$
- with the right action of $SE(2)$ on R^2 :

$$J_{(\xi,u)}(x)=-2\left(\frac{1}{2}\xi\|x\|^2+\Im u.x\right)=-2\left(\frac{1}{2}\|x\|^2,-\Im x\right).(\xi,u)$$

$$J_{(\xi,u)}(x)=J(x).(\xi,u)\Rightarrow J(x)=-2\left(\frac{1}{2}\|x\|^2,-\Im x\right)\ ,\ x\in R^2$$

SOURIAU Cocycle Computation

- We then compute the one-cocycle of $SE(2)$ from the moment map

$$\theta\left(\left(R_{\varphi,\tau}\right)\right)=J\left(0.\left(R_{\varphi},\tau\right)\right)-Ad^{*}_{\left(R_{\varphi},\tau\right)}J(0)=J\left(-R_{-\varphi}\tau\right)$$

$$\theta\left(\left(R_{\varphi,\tau}\right)\right)=-2\left(\frac{1}{2}\left\|\tau\right\|^{2},\Im R_{-\varphi}\tau\right)=-2\left(\frac{1}{2}\left\|\tau\right\|^{2},R_{-\varphi-\frac{\pi}{2}}\tau\right)$$

- Coadjoint orbit of $SE(2)$ are generated by:

$$\mathrm{O}_{(m,\rho)}=\left\{Ad^{*}_{\left(R_{\varphi,\tau}\right)}(m,\rho)+\theta\left(\left(R_{\varphi},\tau\right)\right)/\left(R_{\varphi},\tau\right)\in SE(2)\right\}$$

$$\mathrm{O}_{(m,\rho)}=\left\{\left(x-R_{-\frac{\pi}{2}}\rho.\tau-\left\|\tau\right\|^{2},R_{-\varphi}\rho-2R_{-\varphi-\frac{\pi}{2}}\tau\right)/\left(R_{\varphi},\tau\right)\in SE(2)\right\}$$

FISHER Metric in SOURIAU Model for SE(2)

- The KKS 2-form in non-null cohomology case is given by:

$$\omega_{(m,\rho)(m',\rho')}\left(ad^*_{(\xi,u)}(m',\rho')-(0,2\Im u),ad^*_{(\eta,v)}(m',\rho')-(0,2\Im v)\right)=\rho'.(-\xi\Im v+\eta\Im u)+2u.\Im v$$

$$\text{with}\quad (m',\rho')=\left(x-R_{-\frac{\pi}{2}}\rho.\tau-\left\|\tau\right\|^2,R_{-\varphi}\rho-2R_{-\varphi-\frac{\pi}{2}}\tau\right)\in \mathrm{O}_{(m,\rho)}\subset R^3$$

Souriau Gibbs density for SE(2)

- Considering the symplectic form on R^2

$$\omega(\zeta,\upsilon)=\zeta.\Im\upsilon \quad \text{with} \quad \Im=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- the action of SE(2) is symplectic and admits the momentum map:

$$J(x)=-\left(\frac{1}{2}\|x\|^2,-\Im x\right) \quad , \ x\in R^2$$

- For generalized temperature $\beta\in\Omega=\left\{(b,\mathrm{B})\in se(2)/b<0,\mathrm{B}\in R^2\right\}$, Souriau Gibbs density is given by :

$$p_{Gibbs}(x)=\frac{e^{-\langle J(x),\beta\rangle}}{\int\limits_{R^2} e^{-\langle J(x),\beta\rangle}d\lambda(x)}=\frac{e^{\frac{1}{2}b\|x\|^2-\mathrm{B}.\Im x}}{\int\limits_{R^2} e^{\frac{1}{2}b\|x\|^2-\mathrm{B}.\Im x}d\lambda(x)}$$

Gibbs density for SE(2)

- The Massieu Potential could be computed :

$$\Phi(\beta) = \log \int_{R^2} e^{\frac{1}{2}b\|x\|^2 - \mathrm{B}.\Im x} d\lambda(x) = \log\left(-\frac{2\pi}{b} e^{-\frac{1}{2b}\|\mathrm{B}\|^2} \right)$$

- By derivation of Massieu potential, we can deduce expression of Heat:

$$Q \in \Omega^* = \left\{ (m, M) \in se^*(2) / m + \frac{\|M\|^2}{2} < 0 \right\}; Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \left(\frac{1}{b} - \frac{\|\mathrm{B}\|^2}{2b^2}, \frac{1}{b}\mathrm{B} \right) = \Theta(\beta)$$

- We can the inverse this relation to express generalized temperature with respect to the heat:

$$\beta = \Theta^{-1}(Q) = \left(\left(m + \frac{1}{2}\|M\|^2 \right)^{-1}, \left(m + \frac{1}{2}\|M\|^2 \right)^{-1} M \right)$$

- We can the express the Gibbs density with respect to the Heat Q which is the mean of moment map:

$$p_{Gibbs}(x) = \frac{e^{\frac{\frac{1}{2}\|x\|^2 - M.\Im x}{\left(m + \frac{1}{2}\|M\|^2\right)}}}{\Gamma} \text{ with } \Gamma = \int_{R^2} e^{\frac{\frac{1}{2}\|x\|^2 - M.\Im x}{\left(m + \frac{1}{2}\|M\|^2\right)}} d\lambda(x) \text{ with } (m, M) = E(J(x)) = \left[-E\left(\|x\|^2\right), 2\Im E(x) \right]$$

Gibbs density for SE(2)

Souriau Covariant Gibbs density for SE(2)

$$p_{Gibbs}(x)=\frac{e^{\dfrac{\frac{1}{2}\|x\|^2+2E(x).Ix}{\left(-E\left(\|x\|^2\right)+2\|E(x)\|^2\right)}}}{\displaystyle\int_{R^2} e^{\dfrac{\frac{1}{2}\|x\|^2+2E(x).Ix}{\left(-E\left(\|x\|^2\right)+2\|E(x)\|^2\right)}}d\lambda(x)}$$

Fisher Metric for SE(2)

- Entropy is given by:

$$S(Q)=\langle Q,\beta\rangle-\Phi(\beta)=1+\log(2\pi)+\log\left(-m-\frac{\|M\|^2}{2}\right)$$

- Fisher Metric is given by:

$$I_{Fisher}(Q)=\left(m+\frac{1}{2}\|M\|^2\right)^{-1}\begin{bmatrix} I & M^T \\ M^T & \frac{1}{2}\|M\|^2-m \end{bmatrix}$$

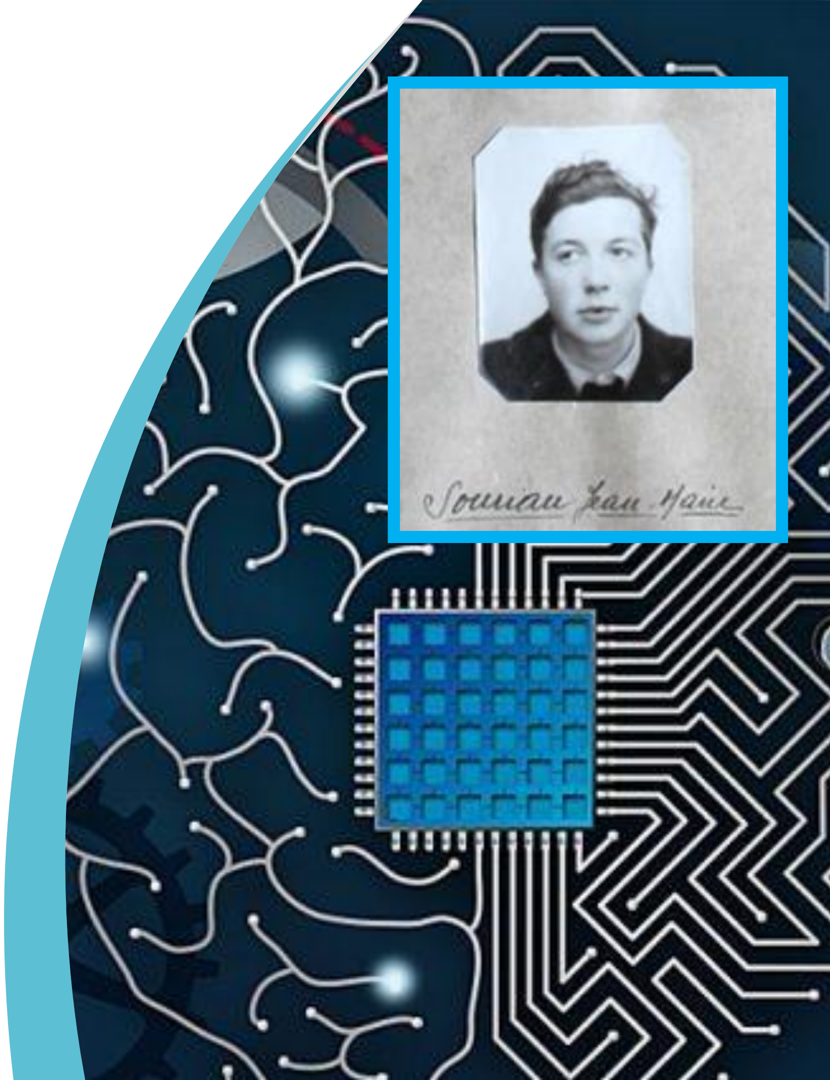
- With $(m,M)=E(J(x))=E\left[-2\left(\frac{1}{2}\|x\|^2,-\Im x\right)\right]=\left[-E\left(\|x\|^2\right),2\Im E(x)\right]$

- Fisher Metric with respect to moments

$$I_{Fisher}(Q)=\left(2\|E(x)\|^2-E\left(\|x\|^2\right)\right)^{-1}\begin{bmatrix} I & (2\Im E(x))^T \\ 2\Im E(x) & 2\|E(x)\|^2+E\left(\|x\|^2\right) \end{bmatrix}$$

THALES

Conclusion & perspectives



Towards Lie Group & Symplectic Machine Learning

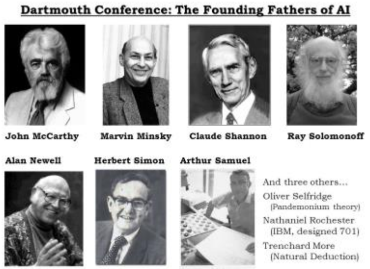
1 <1950

Neural Networks

McCulloch & Pitts (networks of binary neurons can do logic), Donald Hebb (Hebbian synaptic plasticity) Norbert Wiener (cybernetics)

2 1956

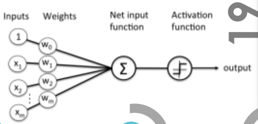
Dartmouth convention



3 1957

Perceptron

Rosenblatt computes learning capabilities



1st winter 60's

4 1990

Natural Gradient

In Information Geometry Framework, RIKEN introduces Natural Gradient based on Fisher Information Matrix (coordinate Free Gradient)

2nd winter 90's

5 2010

Deep Learning

Facebook's AI can perform facial recognition as accurate as human & other developments (Convolutional, Full Convolutional, …)

6 2016

Deep Natural Gradient

LRI ORSAY Lab introduces Robust Natural Gradient for Deep Learning (Yann Ollivier & Gaetan Marceau-Caron)

7 2017

Natural Langevin Dynamics

FACEBOOK & MILA propose Natural Gradient with Stochastic Langevin Gradient (Yann Ollivier & Gaetan Marceau-Caron)

8 2017

Deep Learning On Graph

FACEBOOK & UCLA extend Deep Learning on Graph. Convolutive Network on Graph (Xavier Bresson, …)

9 2018

Deep Learning on Lie Group

Deep network architecture to learn more appropriate Lie group features for 3D action recognition (ETH Zurich, KU Leuven)

Lie Group Machine Learning

Semi-supervised and nuclear learning frameworks based on Lie group Theory (Fanzhang Li, Li Zhang, Zhao Zhang)

2019

Machine Learning on Homogeneous Symplectic Manifold of Lie Group co-adjoint Orbits (Jean-Marie Souriau model)

OPEN

THALES

Supervised & Non-Supervised Learning on Lie Groups

Geodesic Natural Gradient on Lie Algebra

Extension of Neural Network Natural Gradient from Information Geometry on Lie Algebra for Lie Groups Machine Learning

Souriau-Fisher Metric on Coadjoint Orbits

Extension of Fisher Metric for Lie Group through homogeneous Symplectic Manifolds on Lie Group Co-Adjoint Orbits

Souriau Exponential Map on Lie Algebra

Exponential Map for Geodesic Natural Gradient on Lie Algebra based on Souriau Algorithm for Matrix Characteristic Polynomial

Souriau Maximum Entropy Density on Co-Adjoint Orbits

Covariant Maximum Entropy Probability Density for Lie Groups defined with Souriau Moment Map, Co-Adjoint Orbits Method & Kirillov Representation Theory

Fréchet Geodesic Barycenter by Hermann Karcher Flow

Extension of Mean/Median on Lie Group by Fréchet Definition of Geodesic Barycenter on Souriau-Fisher Metric Space, solved by Karcher Flow

Symplectic Integrator preserving Moment Map

Extension of Neural Network Natural Gradient to Geometric Integrators as Symplectic integrators that preserve moment map

Mean-Shift on Lie Groups with Souriau-Fisher Distance

Extension of Mean-Shift for Homogeneous Symplectic Manifold and Souriau-Fisher Metric Space

LIE GROUP SUPERVISED LEARNING

LIE GROUP NON-SUPERVISED LEARNING

Rational to Use Lie Groups for THALES Machine Learning Applications

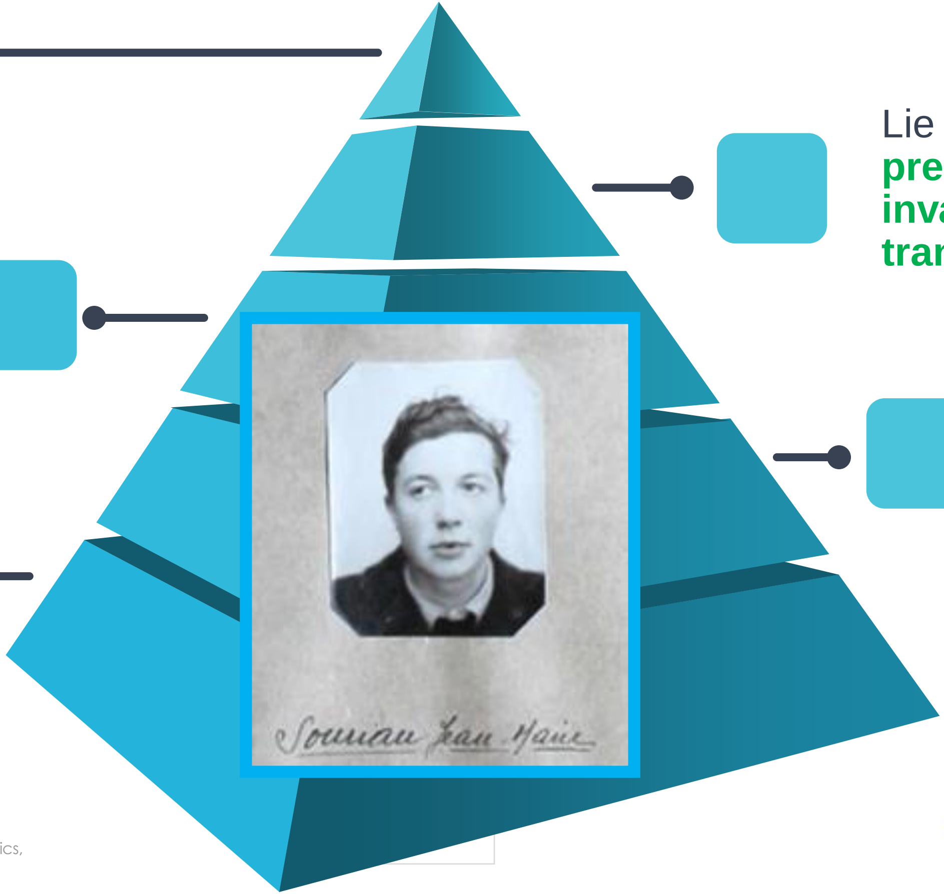
Lie Group is Simple
(natural principles as foundations of Geometry)

Lie Group uses **all Symmetries of your problem**

Lie Group is **Coordinate Free**

Lie Group **preserves invariance wrt all transformations**

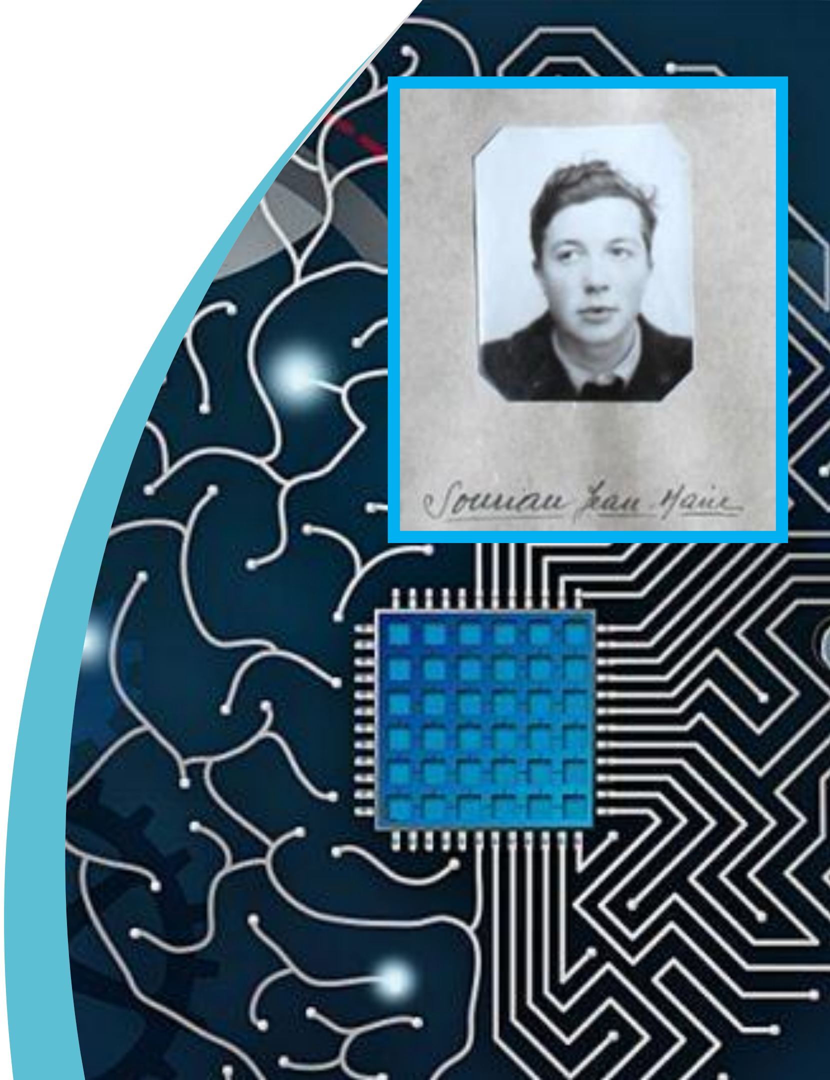
Lie Groups Time Serie Captures **Intrinsic Time Dynamic (e.g. Movement)**



Esprit de finesse et esprit de géométrie

Pour la théorie de la connaissance mais aussi pour les sciences est fondamentale la notion de perspective.
Or, les expériences faites dans la géométrie algébriques, dans la théorie des nombres, et dans l'algèbre abstraite m'induisent à tenter une formulation mathématique de cette notion ***pour surmonter ainsi au moyen de raisonnements d'origine géométrique la géométrie****. Il me semble en effet, que la tendance vers l'abstraction observée dans les mathématiques d'aujourd'hui, loin d'être l'ennemi de l'intuition ait le sens profond de quitter l'intuition pour la faire renaitre dans une alliance entre* ***« esprit de géométrie » et « esprit de finesse », alliance rendue possible par les réserves énormes des mathématiques pures dont Pascal et Goethe ne pouvaient pas encore se douter****.*
Erich Kähler – Sur la théorie des corps purement algébriques, 1952

References



entropy

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Lie Group Machine Learning and Lie Group Structure Preserving Integrators

Guest Editors:

Frédéric Barbaresco
frederic.barbaresco@thalesgroup.com

Prof. Elena Celledoni
elena.celledoni@ntnu.no

Prof. François Gay-Balmaz
francois.gay-balmaz@lmd.ens.fr

Prof. Joël Bensoam
bensoam@ircam.fr

Deadline for manuscript submissions:
6 January 2020

Message from the Guest Editors

Machine/deep learning explores use-case extensions for more abstract spaces as graphs and differential manifolds. Recent fruitful exchanges between geometric science of information and Lie group theory have opened new perspectives to extend machine learning on Lie groups to develop new schemes for processing structured data.

Structure-preserving integrators that preserve the Lie group structure have been studied from many points of view and with several extensions to a wide range of situations. Structure-preserving integrators are numerical algorithms that are specifically designed to preserve the geometric properties of the flow of the differential equation such as invariants, (multi)symplecticity, volume preservation, as well as the configuration manifold. They also naturally find applications in the extension of machine learning and deep learning algorithms to Lie group data.

This Special Issue will collect long versions of papers from contributions presented during the GSI'19 conference, but it will be not limited to these authors and is open to international communities involved in research on Lie group machine learning and Lie group structure-preserving integrators.

mdpi.com/si/30856

Special Issue

Special Issue "Lie Group Machine Learning and Lie Group Structure Preserving Integrators"

Keywords

- Lie groups machine learning
- orbits method
- symplectic geometry
- geometric integrator
- symplectic integrator
- Hamilton's variational principle

https://www.mdpi.com/journal/entropy/special_issues/Lie_group

SOURIAU 2019

J.M. Souriau Interview: https://www.youtube.com/watch?v=uz69vWHXzWY

SOURIAU 2019

- Internet website : http://souriau2019.fr
- In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**
- Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)
- Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES, director of the European ERC)

Jean-Marie SOURIAU
and
Symplectic Geometry

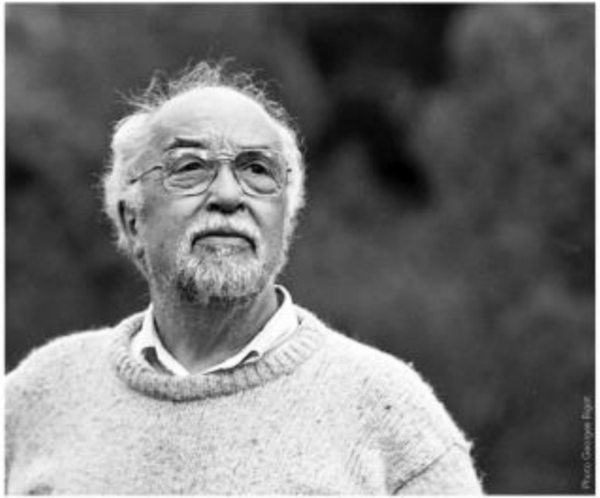
Jean-Pierre BOURGUIGNON
(CNRS-IHÉS)

https://www.youtube.com/watch?v=93hFolIBo0Q&t=3s

SOURIAU 2019

Conference May 27-31 2019, Paris-Diderot University

https://www.youtube.com/watch?v=beM2pUK1H7o



Frédéric Barbaresco
Daniel Bennequin
Jean-Pierre Bourguignon
Pierre Cartier
Dan Christensen
Maurice Courbage
Thibault Damour
Paul Donato
Paolo Giordano
Serap Gürer
Patrick Iglesias-Zemmour
Yael Karshon
Jean-Pierre Magnot
Yvette Kosmann-Schwarzbach
Marc Lachièze-Rey
Martin Pinsonnault
Elisa Prato
Urs Schreiber
Jean-Jacques Szczeciniarz
Roland Triay
Jordan Watts
Enxin Wu
San Vũ Ngọc
Alan Weinstein

JEAN-MARIE SOURIAU

In 1969, the groundbreaking book of Jean-Marie Souriau appeared "Structure des Systèmes Dynamiques". We will celebrate, in 2019, the jubilee of its publication, with a conference in honour of the work of this great scientist.

Symplectic Mechanics, Geometric Quantization, Relativity, Thermodynamics, Cosmology, Diffeology & Philosophy

80|Prime

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FGSI'19 Cartan-Koszul-Souriau
Foundations of Geometric Structures of Information

Foundations of Geometric Structures of Information

4-6 Feb 2019 Montpellier (France)

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MAIN MENU

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- Speakers
- Planning
- Registration
- List of Participants
- Documents of the conference
- Practical information
- Sponsors

HELP

PRESENTATION

A seminar on Topological and Geometrical Structures of Information has been organized at CIRM in 2017, to gather engineers, applied and pure mathematicians interested in the geometry of information. This year FGSI'19 conference will be focused on the foundations of geometric structures of information. It is dedicated to the triumvirat Cartan - Koszul - Souriau and their influence on the field.

Poster

https://fgsi2019.sciencesconf.org/

La Région Occitanie Pyrénées - Méditerranée

MUSE MONTPELLIER UNIVERSITY OF EXCELLENCE

IMAG

cnrs

Anton ALEKSEEV (Geneva Univ.)
Dmitri ALEKSEEVSKY (Moscow IITP)
John BAEZ (Riverside UC)
Michel BRION (Grenoble Univ.)
Misha GROMOV (Paris IHES)
Patrick IGLESIAS-ZEMMOUR (Aix-Marseille Univ.)
Yann OLLIVIER (Paris Facebook)
Vasily PESTUN (Paris IHES)
Aissa WADE (Penn State Univ.)

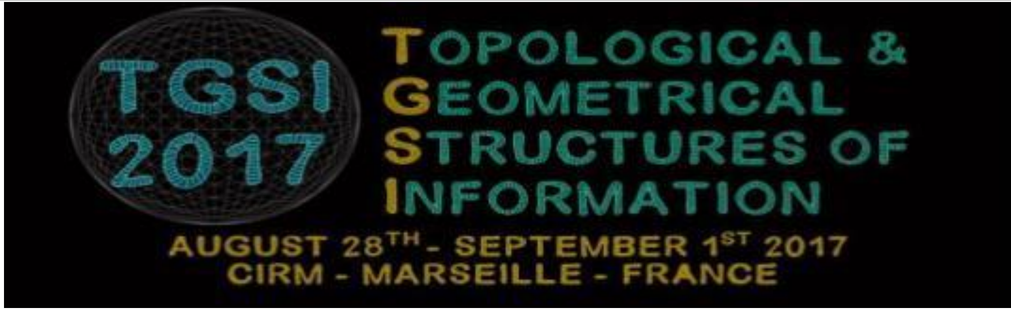
Panel sessions: SYMPLECTIC GEOMETRY IN PHYSICS
TRIBUTE TO J-L KOSZUL & J-M SOURIAU

CIRM Seminar, August 2017
TGSI'17 « Topological & Geometrical Structures of Information »

TGSI'17 videos & slides
http://forum.cs-dc.org/category/94/tgsi2017

Special Issue "**Topological and Geometrical Structure of Information**", Selected Papers from CIRM conferences 2017"

http://www.mdpi.com/journal/entropy/special_issues/topological_geometrical_info

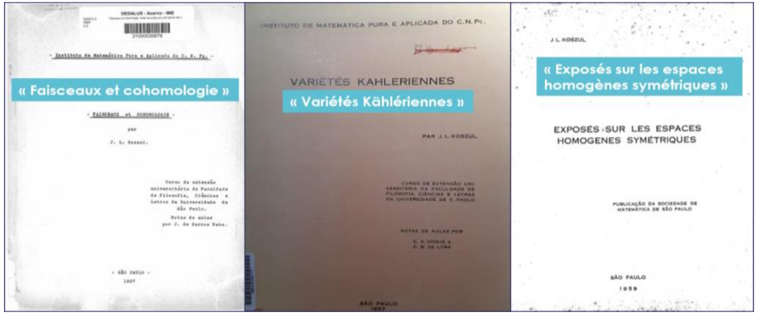


Talk on Koszul-Souriau Characteristic Function:
https://www.youtube.com/watch?v=VXxiMCn-tsE&feature=youtu.be

Jean-Louis Koszul was
foreign member of **São Paulo Academia of Sciences**

Jean Louis Koszul Lectures at Sao Paulo:

- Faisceaux et Cohomologie
- Variétés Kählériennes
- Exposés sur les espaces homogènes symétriques



Sao Paulo Journal of Mathematical Sciences
SPRINGER
Editor-in-Chief: **Claudio Gorodski**
https://www.springer.com/mathematics/journal/40863

OPEN

GSI'13 Mines ParisTech

Slides :
https://www.see.asso.fr/gsi2013

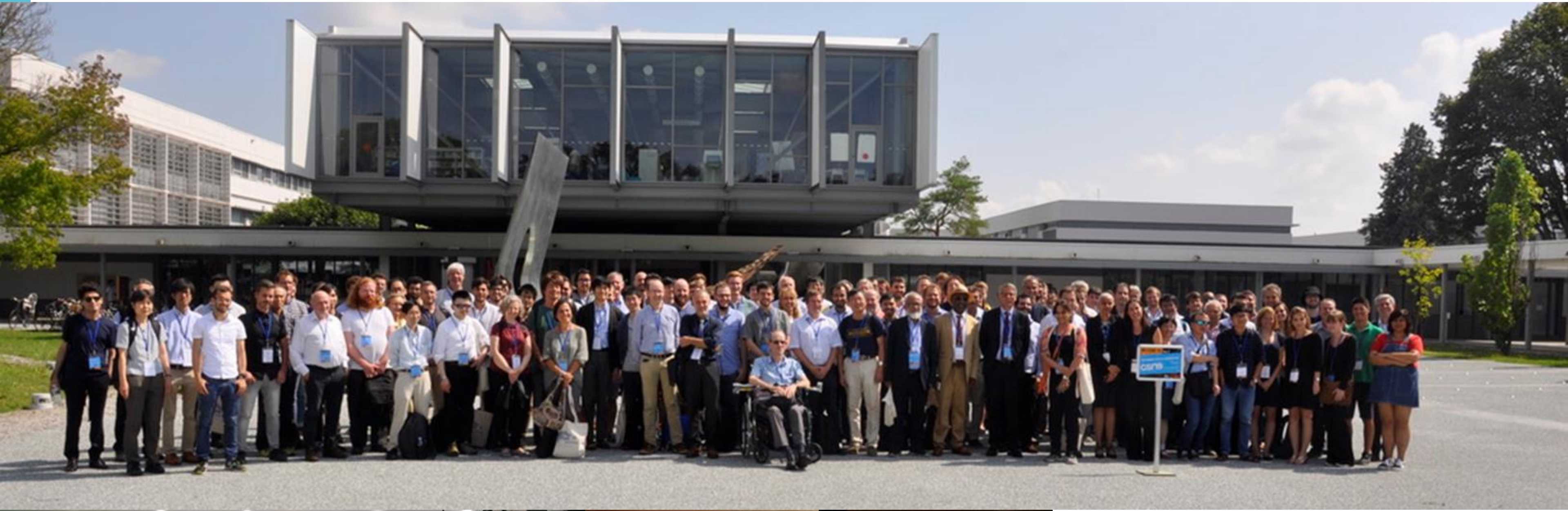
GSI'15 Ecole Polytechnique

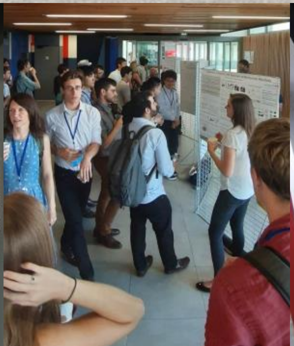
Videos:
https://www.youtube.com/channel/UC5HHo1jbQXusNQzU1iekaGA
UNITWIN website (slides & videos):
http://forum.cs-dc.org/category/90/gsi2015

GSI'17 Mines ParisTech

Videos: https://www.youtube.com/channel/UCnE9-LbfFRqtaes49cN2DVg/videos
UNITWIN website (slides & videos):
http://forum.cs-dc.org/category/135/gsi2017

GSI'19 ENAC





website :
https://www.see.asso.fr/en/GSI2019

THALES

Fisher Metric by Misha Gromov (IHES)

M. Gromov, In a Search for a Structure, Part 1: On Entropy. July 6, 2012

- http://www.ihes.fr/~gromov/PDF/structre-serch-entropy-july5-2012.pdf

Gromov Six Lectures on Probability, Symmetry, Linearity. October 2014, Jussieu, November 6th , 2014

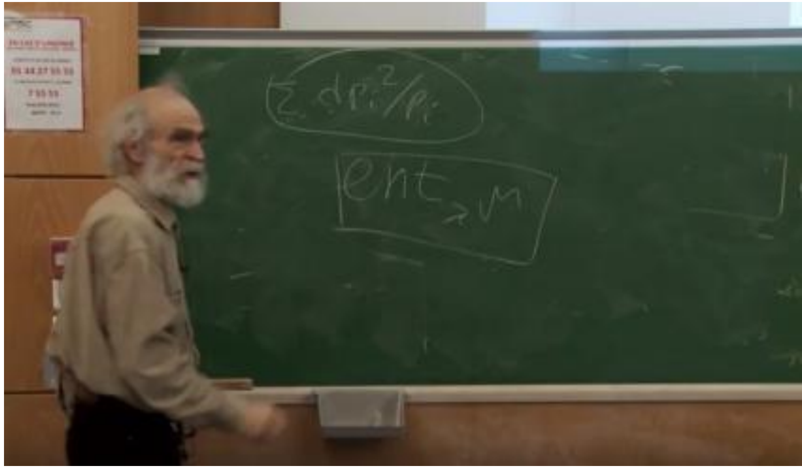
- Lecture Slides & video:

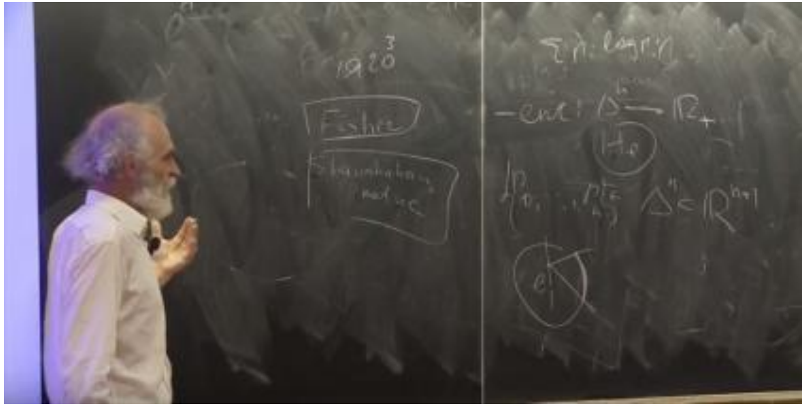
http://www.ihes.fr/~gromov/PDF/probability-huge-Lecture-Nov-2014.pdf

https://www.youtube.com/watch?v=hb4D8yMdov4

Gromov Four Lectures on Mathematical Structures arising from Genetics and Molecular Biology, IHES, October 2013

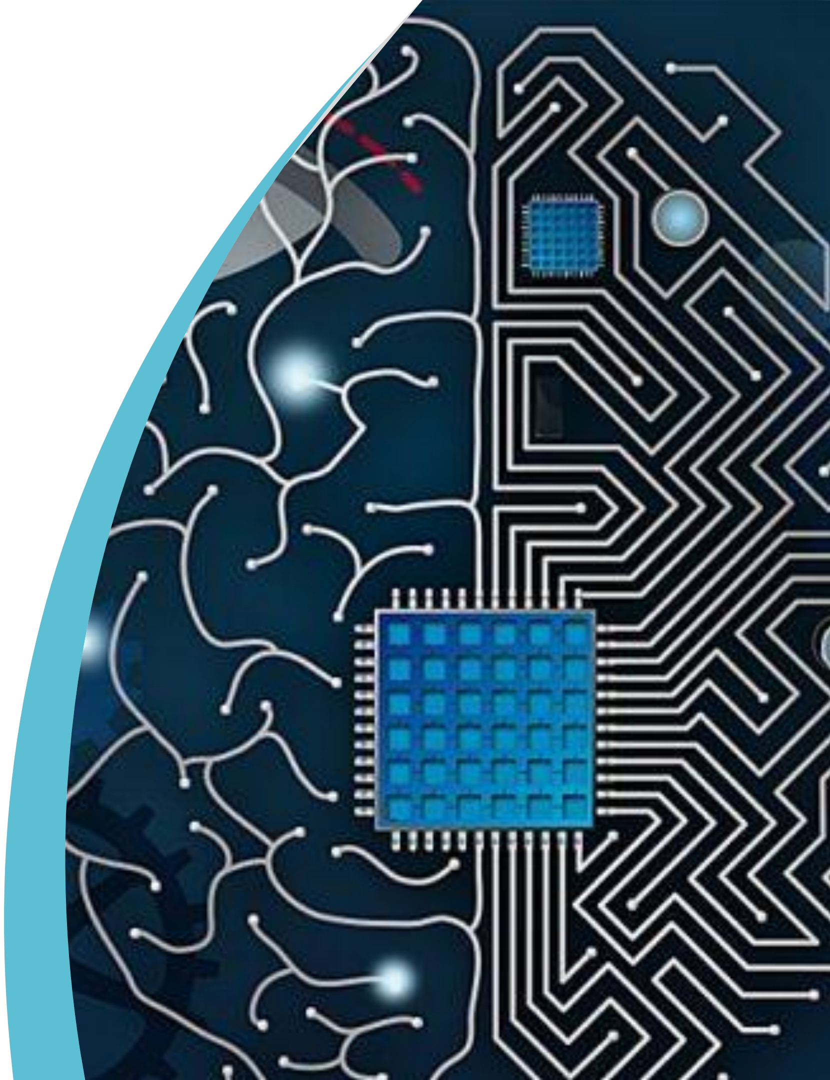
https://www.youtube.com/watch?v=v7QuYuoyLQc&t=5935s
(at time 01h35min)





THALES

Main references



Koszul Book on Souriau Work

Jean-Louis Koszul · Yiming Zou

Introduction to Symplectic Geometry

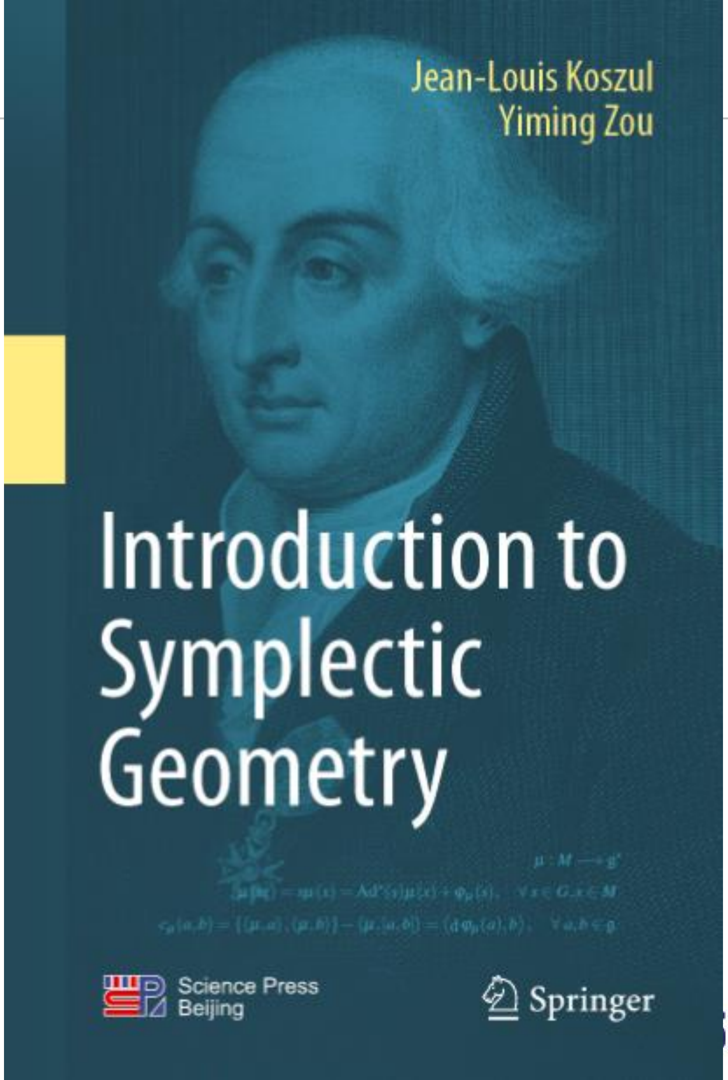
Forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle

This introductory book offers a unique and unified overview of symplectic geometry, highlighting the differential properties of symplectic manifolds. It consists of six chapters: Some Algebra Basics, Symplectic Manifolds, Cotangent Bundles, Symplectic G-spaces, Poisson Manifolds, and A Graded Case, concluding with a discussion of the differential properties of graded symplectic manifolds of dimensions (o,n). It is a useful reference resource for students and researchers interested in geometry, group theory, analysis and differential equations.

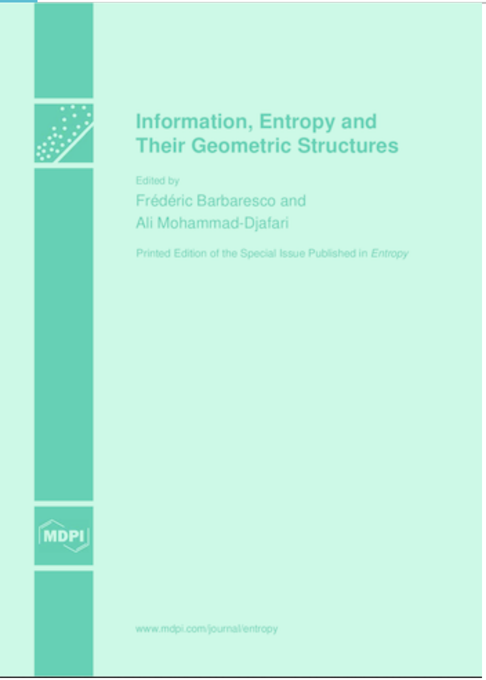
$$\mu : M \longrightarrow \mathfrak{g}^*$$

$$\mu(sx) = s\mu(x) = \mathrm{Ad}^*(s)\mu(x) + \varphi_\mu(s), \quad \forall s \in G, x \in M$$

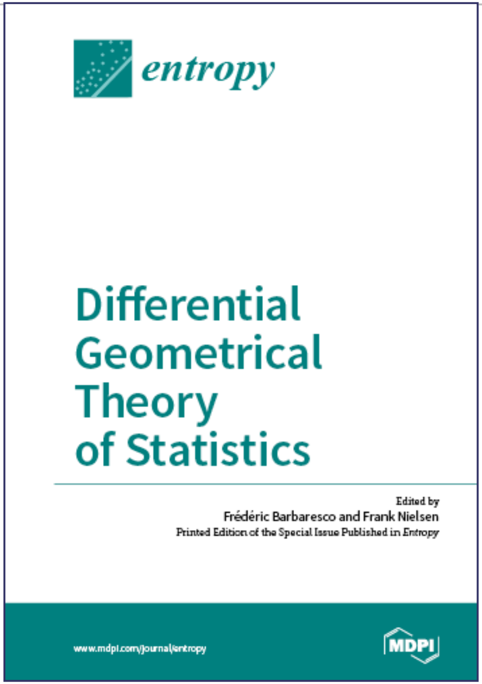
$$c_\mu(a,b) = \{\langle \mu, a\rangle, \langle \mu, b\rangle\} - \langle \mu, [a,b]\rangle = \langle d\varphi_\mu(a), b\rangle, \quad \forall a, b \in \mathfrak{g}$$



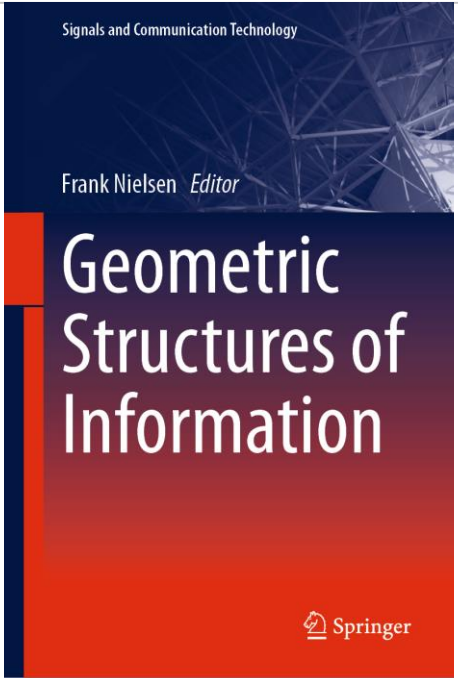
Books



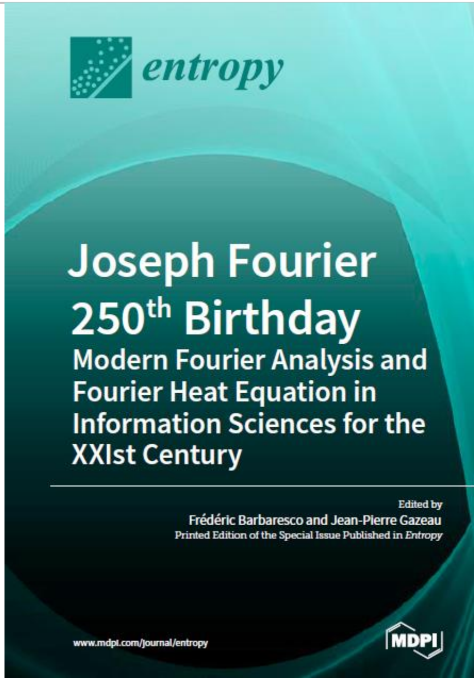
https://www.mdpi.com/books/pdfview/book/127



https://www.mdpi.com/books/pdfview/book/313

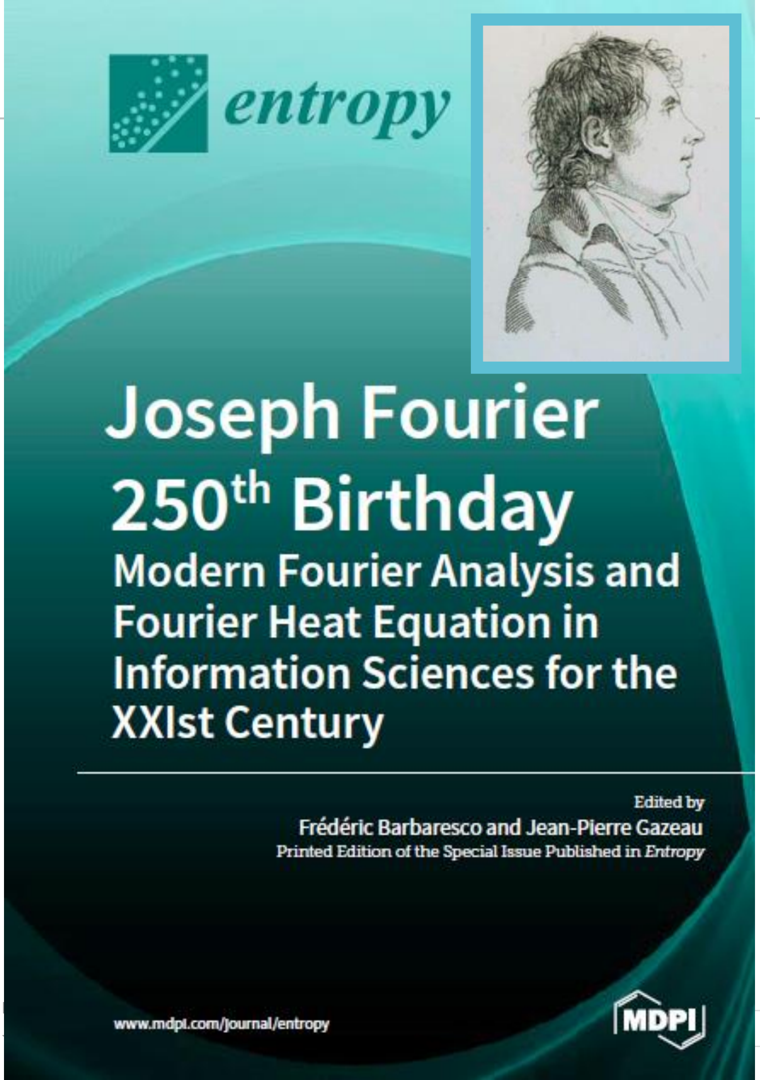


https://www.springer.com/us/book/9783030025199



https://www.mdpi.com/journal/entropy/special_issues/fourier

Jean-Marie Souriau Geometric Theory of Heat, 250 years after Joseph Fourier



MDPI Entropy Book for Joseph Fourier 250th Birthday

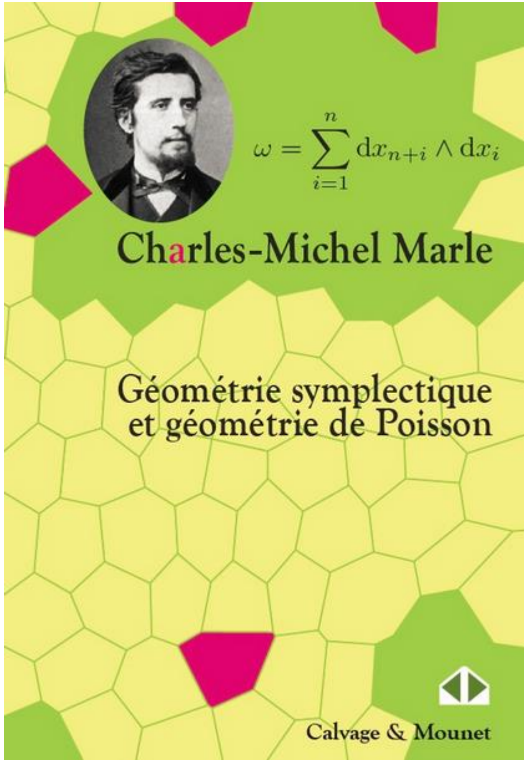
- https://www.mdpi.com/journal/entropy/special_issues/fourier

Jean-Marie Souriau Geometric Theory of Heat: Bedrock for Lie Group Machine Learning

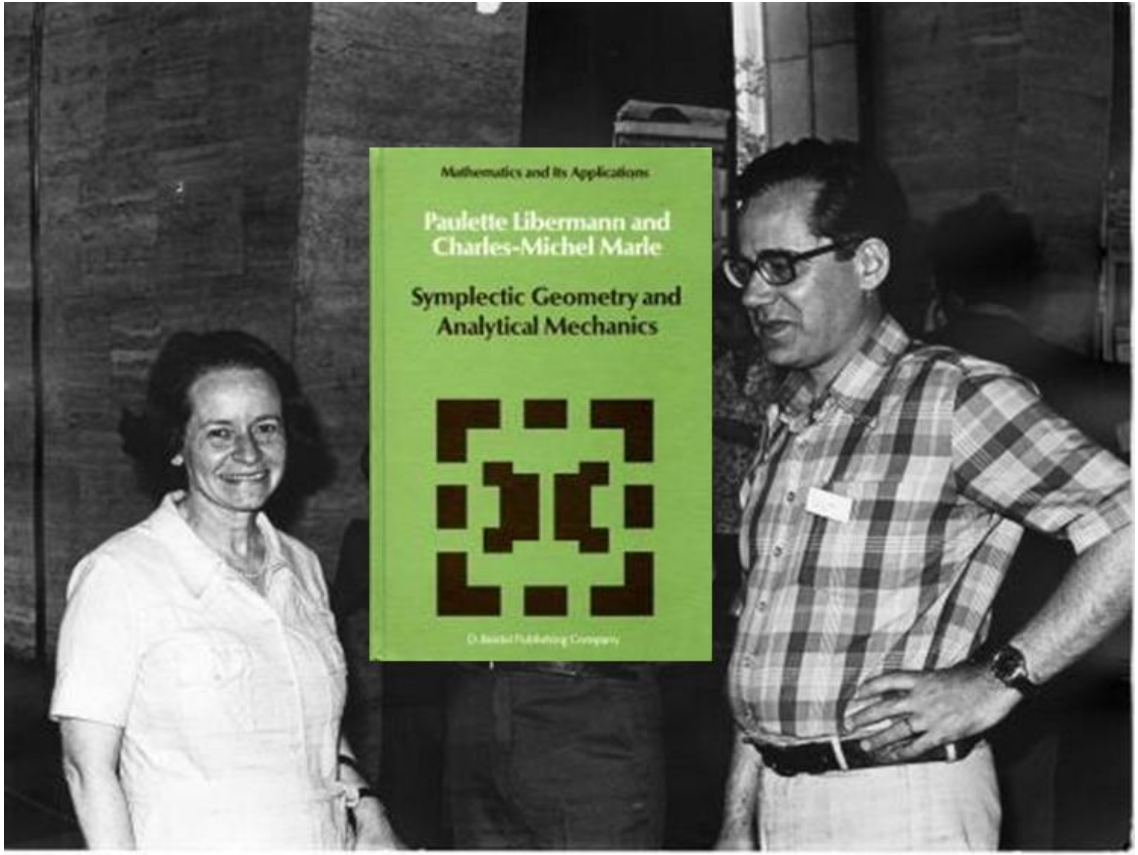
- Barbaresco, F. : Higher Order Geometric Theory of Information and Heat Based on Poly-Symplectic Geometry of Souriau Lie Groups Thermodynamics and Their Contextures: The Bedrock for Lie Group Machine Learning. Entropy, 20, 840, (2018).

FOURIER 250.

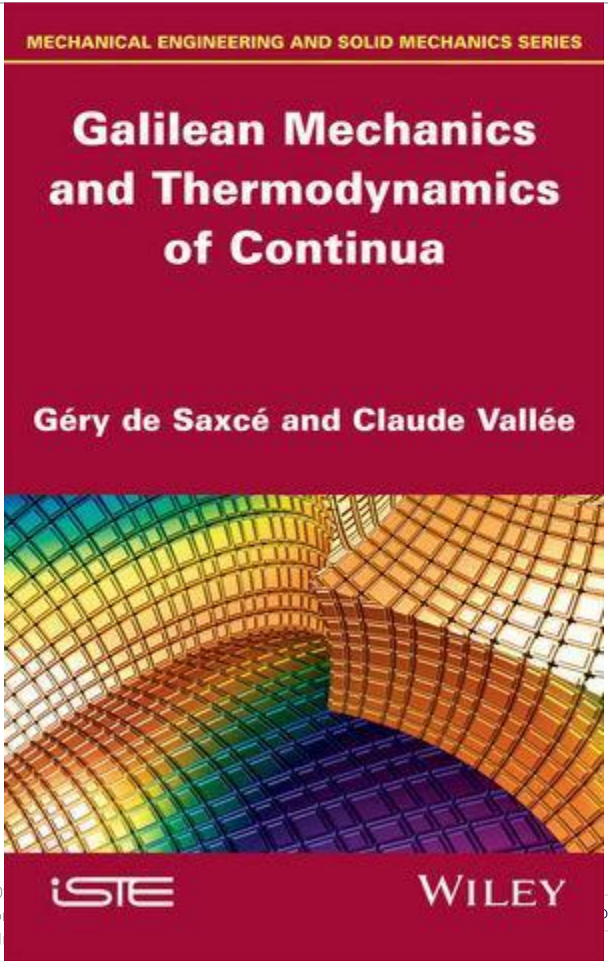
Charles-Michel Marle Books



https://www.amazon.fr/gp/product/2916352708/ref=dbs_a_def_rwt_bibl_vppi_i0



Référence Book: Gery de Saxcé & Claude Vallée



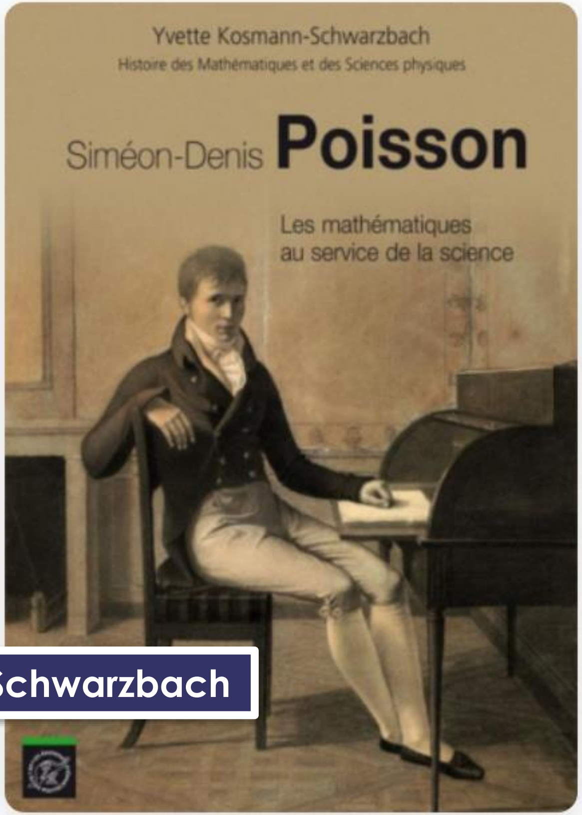
This title proposes a unified approach to continuum mechanics which is consistent with Galilean relativity. Based on the notion of affine tensors, a simple generalization of the classical tensors, this approach allows gathering the usual mechanical entities — mass, energy, force, moment, stresses, linear and angular momentum — in a single tensor.

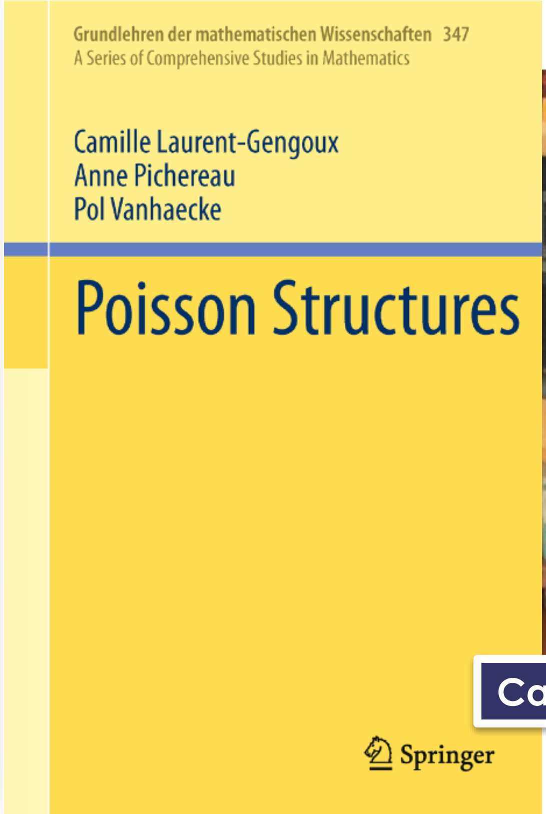
Starting with the basic subjects, and continuing through to the most advanced topics, the authors' presentation is progressive, inductive and bottom-up. They begin with the concept of an affine tensor, a natural extension of the classical tensors. The simplest types of affine tensors are the points of an affine space and the affine functions on this space, but there are more complex ones which are relevant for mechanics – torsors and momenta. The essential point is to derive the balance equations of a continuum from a unique principle which claims that these tensors are affine-divergence free.

https://www.wiley.com/en-us/Galilean+Mechanics+and+Thermodynamics+of+Continua-p-9781848216426

Poisson Geometry

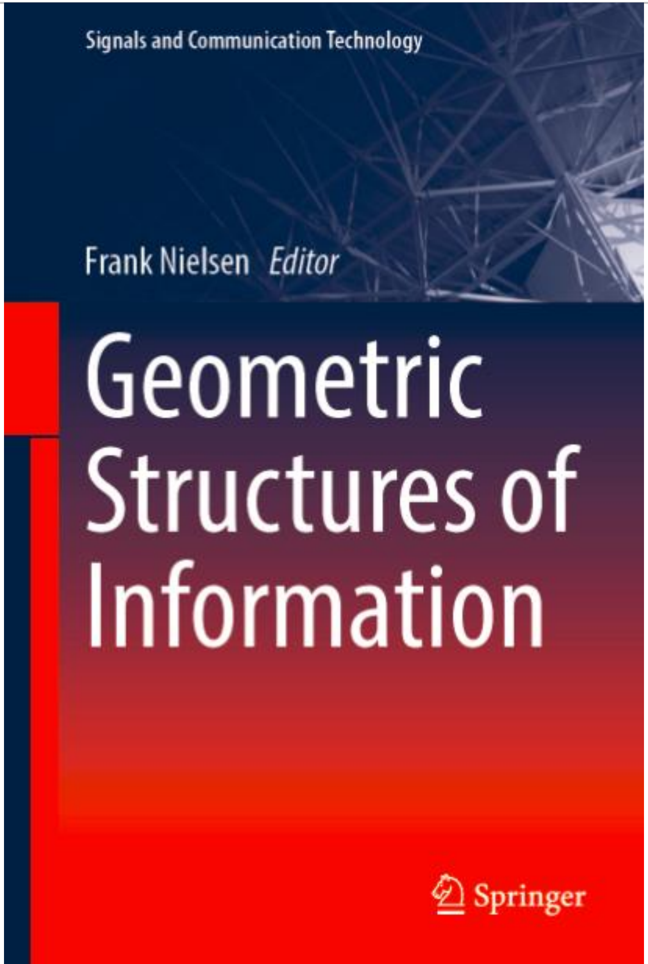
Yvette Kosmann-Schwarzbach





Camille Laurent-Gengoux

Geometric Structures of Information, SPRINGER



Geometric Structures of Information

- https://www.springer.com/us/book/9783030025199

Paper on Jean-Louis Koszul

- Barbaresco, F. , Jean-Louis Koszul and the Elementary Structures of Information Geometry, Geometric Structures of Information, pp 333-392, SPRINGER, 2018
- https://link.springer.com/chapter/10.1007%2F978-3-030-02520-5_12

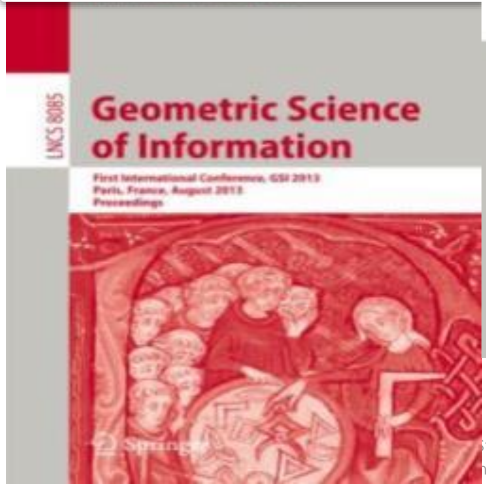
GSI SPRINGER PROCEEDINGS Collection

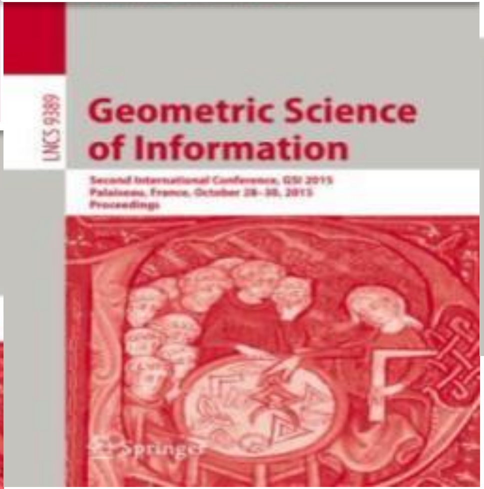
GSI'19 Springer Proceedings: https://www.springer.com/gp/book/9783030269791

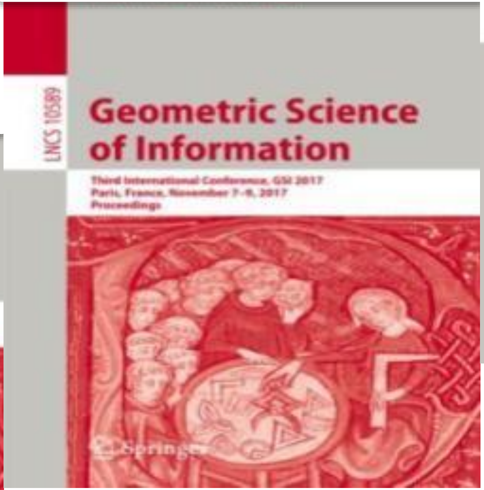
GSI'17 Springer Proceedings: http://www.springer.com/cn/book/9783319684444

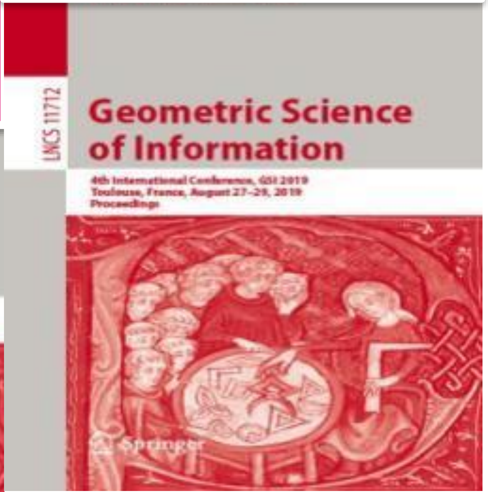
GSI'15 Springer Proceedings: http://www.springer.com/la/book/9783319250397

GSI'13 Springer Proceedings: http://www.springer.com/us/book/9783642400193









Seminal Work of Muriel Casalis supervised by Gérard Letac

International Statistical Review (1991), 59, 2, pp. 241–262. Printed in Great Britain

Familles Exponentielles Naturelles sur $\mathbb{R}^d$ Invariantes par un Groupe

Muriel Casalis

Laboratoire de Statistique, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex, France

Résumé

La caractérisation des familles exponentielles naturelles de $\mathbb{R}^d$ préservées par un groupe d'affinités donné est faite dans trois cas: celui d'un groupe compact, en particulier du groupe des rotations, celui du groupe hyperbolique et enfin celui d'un groupe quelconque de translations. La démarche adoptée consiste à traduire la propriété d'invariance de la famille par une propriété portant sur les mesures qui l'engendrent puis à rechercher ces dernières en conséquence.

1 Introduction

Il est courant en statistique d'envisager un modèle $(\Omega, \mathscr{A}, (P_\theta)_{\theta\in\Theta})$ tel qu'il existe un groupe G de permutations de Ω préservant globalement la famille de probabilités $F = \{P_\theta, \theta \in \Theta\}$, c'est-à-dire que pour tout (θ, g) de $\Theta \times G$, l'image $g(P_\theta)$ de P_θ par g est encore dans F (Barndorff-Nielsen parle alors de modèle de transformations). On pourra consulter Barndorff-Nielsen et al. (1982) et plus récemment le livre de Barndorff-Nielsen (1988).

Un exemple célèbre est celui des distributions de Fisher–Von-Mises pour lequel Ω est la sphère unité de l'espace euclidien E,

$$P_\theta(dx) = L(\theta)^{-1} \exp\langle\theta, x\rangle \sigma(dx),$$

σ désignant la probabilité uniforme sur Ω et $L(\theta)$ le coefficient de normalisation, et pour lequel G est le groupe des rotations $\mathbb{O}(E)$ de E.

Dans cet exemple, $\{P_\theta, \theta \in \Theta\}$ est une famille exponentielle naturelle au sens suivant.

Soit E un espace vectoriel de dimension finie, E^* son dual et si (θ, x) est dans $E^* \times E$, $\langle\theta, x\rangle$ désigne le crochet de dualité; soit, de plus, μ une mesure de Radon positive sur E; on note alors L_μ la transformée de Laplace de μ définie par:

$$L_\mu : E^* \to [0, \infty]: \theta \to \int_E \exp\langle\theta, x\rangle\mu(dx);$$

D_μ est l'ensemble $\{\theta \in E^*, L_\mu(\theta) < \infty\}$, $\Theta(\mu)$ son intérieur et k_μ la fonction définie sur $\Theta(\mu)$ par:

$$k_\mu(\theta) = \text{Log}\, L_\mu(\theta). \tag{1.1}$$

On désigne aussi par $\mathscr{M}(E)$ l'ensemble des mesures de Radon μ positives telles que:

(i) μ n'est pas concentrée sur un sous-espace affine strict de E;
(ii) $\Theta(\mu)$ est non vide.

N° d'ordre 679

THESE

présentée à

L'UNIVERSITE PAUL SABATIER DE TOULOUSE (SCIENCES)

pour obtenir

DOCTORAT DE L'UNIVERSITE PAUL SABATIER

Spécialité : MATHEMATIQUES APPLIQUEES

par

Muriel BONNEFOY - CASALIS

—

FAMILLES EXPONENTIELLES NATURELLES

INVARIANTES PAR UN GROUPE

—

Soutenue le 11 Juin 1990, devant la Commission d'Examen :

MM.	H. CAUSSINUS	Professeur à l'Université Paul Sabatier
	D. BAKRY	Professeur à l'Université Paul Sabatier
	J. FARAUT	Professeur à l'Université PARIS VI
	Y. GUIVARC'H	Professeur à l'Université PARIS VI
	G. LETAC	Professeur à l'Université Paul Sabatier

Laboratoire de Statistique et Probabilités
UNIVERSITE PAUL SABATIER

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Covariant Gibbs Density by Souriau Thermodynamics

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THALES

Souriau Entropy Definition

Entropy Definition by Jean-Marie Souriau (1/4)

- Let E a vector space of finite size, μ a measure on the dual space E^*, then the function given by:

$$\alpha \mapsto \int_{E^*} e^{M\alpha} \mu(M) dM$$

for all $\alpha \in E$ such that the integral is convergent.

- This function is called **Laplace Transform**. This transform F of the measure μ is differentiable inside its definition set $def(F)$. Its p-th derivables are given by the following convergent integrals :

$$F^{(p)}(\alpha) = \int_{E^*} M \otimes M \ldots \otimes M \mu(M) dM$$

Entropy Definition by Jean-Marie Souriau (2/4)

Souriau Theorem:

- Let E a vector space of finite size, μ a non zero positive measure of its dual space E^* , F its Laplace transform, then:
  - F is a semi-definite convex function, $F(\alpha) > 0, \forall \alpha \in def(F)$
  - $f = \log(F)$ is convex and semi-continuous
  - Let α an interior point of $def(F)$ then:
    - $D^2(f)(\alpha) \geq 0$
    - $D^2(f)(\alpha) = \int_{E^*} e^{M\alpha}[M - D(f)(\alpha)]^{\otimes^2} \mu(M)dM$
    - $D^2(f)(\alpha)$ inversible $\Leftrightarrow$ affine Enveloppe (support(μ)) $=E^*$

Entropy Definition by Jean-Marie Souriau (3/4)

Lemme:

- Let X a locally compact space, Let λ a positive measure of X , with X as support, then the following function Φ is convex:

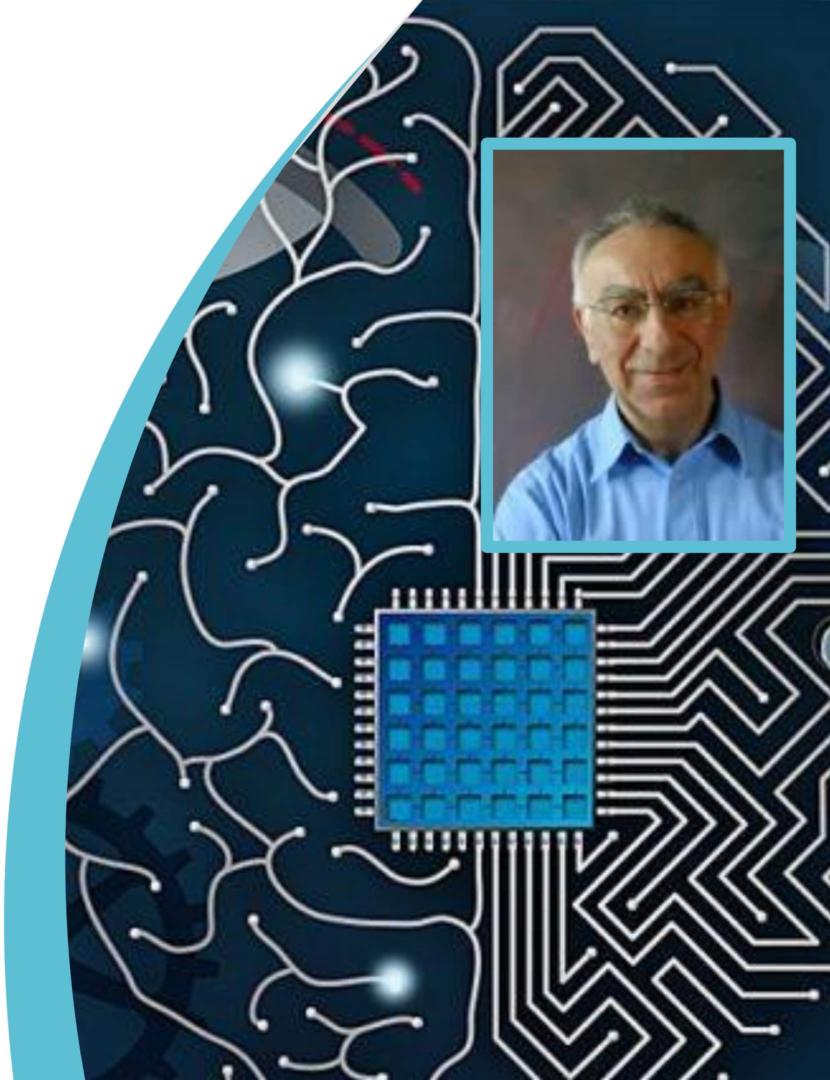
$$\Phi(h) = \log \int_X e^{h(X)} \lambda(x) dx \ , \ \forall h \in C(X)$$

such that the integral is convergent.

Proof:

- The integral is strictly positive when its converges, insuring existence of its logarithm
- Epigraph Φ is the set of $\begin{pmatrix} h \\ y \end{pmatrix}$ such that $\int_X e^{h(x)-y} \lambda(x) dx \leq 1$.
- Convexity of exponential prove that this epigraph is convex.

Balian Computation of Gibbs Density for Dynamical Centrifuge System



Roger Balian Computation of Gibbs density for centrifuge

- Balian has computed the Boltzmann-Gibbs distribution without knowing Souriau equations. Exercice 7b of :
  - **Balian, R. From Microphysics to Macrophysics, 2nd ed.; Springer: Berlin, Germany, 2007; Volume I**
- Balian started by considering the constants of motion that are the energy and the component J_z of the total angular momentum: $J=\sum_i (r_i \times p_i)$
- Balian observed that he must add to the Lagrangian parameter, given by (Planck) temperature β for energy, an additional one associated with J_z . He identifies this additional multiplier with $-\beta\omega$ by evaluating the mean velocity at each point.
- He then introduced the same results also by changing the frame of reference, the Lagrangian and the Hamiltonian in the rotating frame and by writing down the canonical equilibrium in that frame. He uses the resulting distribution to find, through integration, over the momenta, an expression for the particles density as the function of the distance from the cylinder axis.

Roger Balian Computation of Gibbs density for centrifuge

- The fluid carried along by the walls of the rotating vessel acquires a non-vanishing average angular momentum $\langle J_z \rangle$ around the axis of rotation, that is a constant of motion. In order to be able to assign to it a definite value, Balian proposed to associate with it a Lagrangian multiplier λ , in exactly the same way as we classically associate the multiplier β with the energy in canonical equilibrium. The average $\langle J_z \rangle$ will be a function of λ . The Gibbs density for rotating gas is given by Balian as:

$$D = \frac{1}{Z} e^{-\beta H - \lambda J_z} = \frac{1}{Z} \exp\left\{ \sum_i \left[\frac{\beta p_i^2}{2m} + \lambda \left(x_i p_{y_i} - y_i p_{x_i} \right) \right] \right\}$$

- With the energy and the average angular momentum given by:

$$U = -\frac{\partial \ln Z}{\partial \beta} = \frac{1}{kT} \qquad \langle J_z \rangle = -\frac{\partial \ln Z}{\partial \lambda}$$

Roger Balian Computation of Gibbs density for centrifuge

- The Lagrangian parameter λ has a mechanical nature. To identify this parameter, Balian compared microscopic and macroscopy descriptions of fluid mechanics. He described the single-particle reduced density by:

$$f(r,p) \propto \exp\left\{-\frac{\beta p^2}{2m} - \lambda\left(xp_y - yp_x\right)\right\} = \exp\left\{-\frac{\beta}{2m}\left(p + \frac{m}{\beta}[\lambda \times r]\right)^2 + \frac{m\lambda^2}{2\beta}\left(x^2 + y^2\right)\right\}$$

- Whence Balian finds the velocity distribution at a point r to be proportional to:

$$\exp\left\{-\frac{m}{2kT}\left(v + \frac{1}{\beta}[\lambda \times r]\right)^2\right\}$$

- The mean velocity of the fluid at the point r is equal to: $\langle v \rangle = -\frac{1}{\beta}[\lambda \times r]$

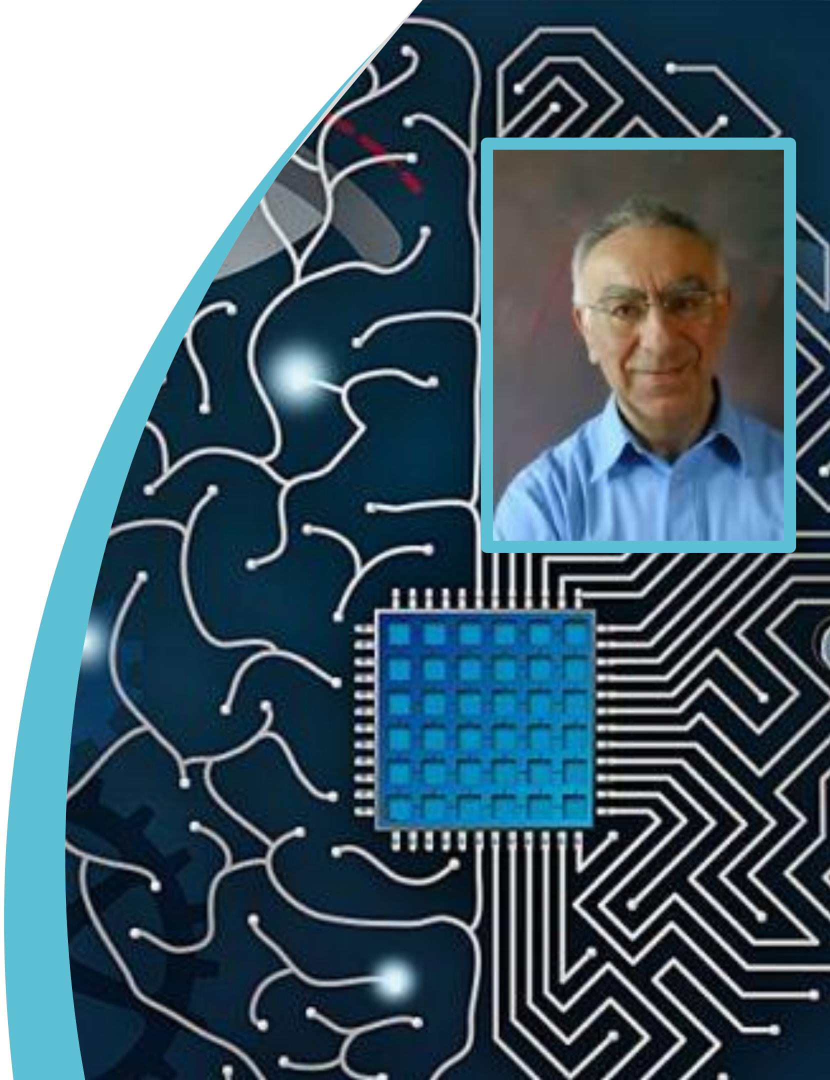
- and can be identified with the velocity $[\omega \times r]$ in an uniform rotation with angular velocity ω. By comparison, Balian put : $\omega = -\lambda / \beta$

Roger Balian Computation of Gibbs density for centrifuge

- Balian made the remarks that ***“The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”.***

THALES

Compliance with Symplectic Model of Thermodynamics by Balian-Valentin



Compatible Balian Gauge Theory of Thermodynamics

- Entropy S is an extensive variable $q^0 = S(q^1,\ldots,q^n)$ depending on $q^i \ (i=1,\ldots,n)$ n independent extensive/conservative quantities characterizing the system

- The n intensive variables γ_i are defined as the partial derivatives:

$$\gamma_i = \frac{\partial S(q^1,\ldots,q^n)}{\partial q^i}$$

- Balian has introduced a non-vanishing gauge variable which multiplies all the intensive variables, defining a new set of variables: $p_i = -p_0.\gamma_i \quad , \quad i=1,\ldots,n$

- The 2n+1-dimensional space is thereby extended into a 2n+2-dimensional thermodynamic space T spanned by the variables p_i, q^i with $i=0,1,\ldots,n$,where the physical system is associated with a n+1-dimensional manifold M in T, parameterized for instance by the coordinates $q^1,\ldots,q^n$ and p_0 .

Compatible Balian Gauge Theory of Thermodynamics

- the contact structure in 2n+1 dimension: $\tilde{\omega} = dq^0 - \sum_{i=1}^{n} \gamma_i . dq^i$
- is embedded into a symplectic structure in 2n+2 dimension, with 1-form, as symplectization: $\omega = \sum_{i=0}^{n} p_i . dq^i$
- The n +1-dimensional thermodynamic manifolds M are characterized by : $\omega = 0$. The 1-form induces then a symplectic structure on T : $d\omega = \sum_{i=0}^{n} dp_i \wedge dq^i$
- The concavity of the entropy $S(q^1,...,q^n)$, as function of the extensive variables, expresses the stability of equilibrium states. It entails the existence of a metric structure in the n-dimensional space q_i :

$$ds^2 = -d^2S = -\sum_{i,j=1}^{n} \frac{\partial^2 S}{\partial q^i \partial q^j} dq^i dq^j$$

- which defines a distance between two neighboring thermodynamic states:

$$d\gamma_i = \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q^i \partial q^j} dq^j$$

$$ds^2 = -\sum_{i=1}^{n} d\gamma_i dq_i = \frac{1}{p_0} \sum_{i=0}^{n} dp_i dq^i$$

Compatible Balian Gauge Theory of Thermodynamics

- We can observe that this Gauge Theory of Thermodynamics is compatible with Souriau Lie Group Thermodynamics, where we have to consider the Souriau vector :

$$\beta = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$ transformed in a new vector $p_i = -p_0 . \gamma_i$

$$p = \begin{bmatrix} -p_0\gamma_1 \\ \vdots \\ -p_0\gamma_n \end{bmatrix} = -p_0 . \beta$$

Souriau Model
& Multivariate Gaussian Model

Multivariate Gaussian Density as 1st order Maximum Entropy in Souriau Book (Chapter IV)

Exemple : (*loi normale*) :

Prenons le cas $V = R^n$, λ = mesure de Lebesgue, $\Psi(x) \equiv \begin{pmatrix} x \\ x \otimes x \end{pmatrix}$;

un élément Z du dual de E peut se définir par la formule

$$Z(\Psi(x)) \equiv \bar{a}.x + \tfrac{1}{2}\bar{x}.H.x$$

[$a \in R^n$; H = matrice symétrique]. On vérifie que la convergence de l'intégrale I_0 a lieu si la matrice H est positive ([1]) ; dans ce cas la loi de Gibbs s'appelle *loi normale de Gauss* ; on calcule facilement I_0 en faisant le changement de variable $x^* = H^{1/2}x + H^{-1/2}a$ ([2]) ; il vient

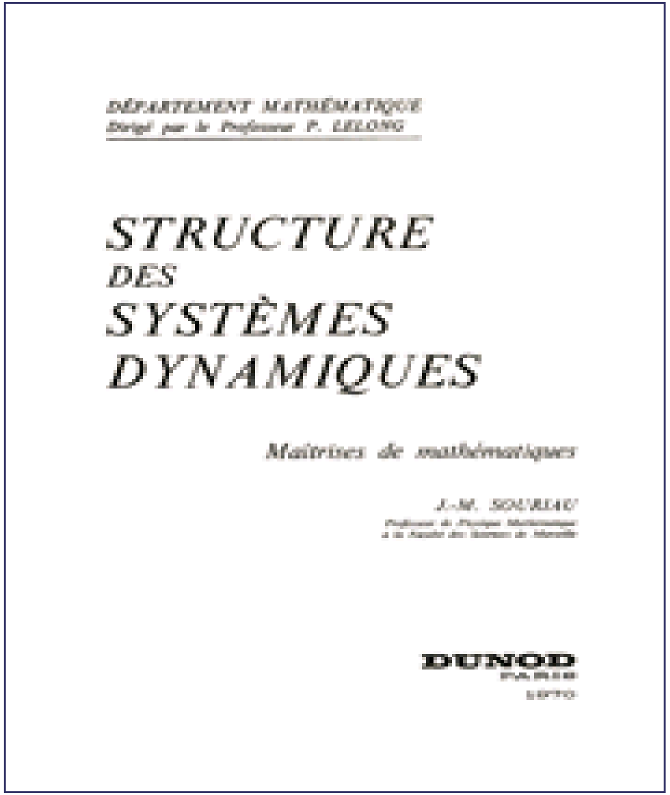
$$\boxed{z = \tfrac{1}{2}\left[\bar{a}.H^{-1}.a - \log(\text{dét}(H)) + n\log(2\pi)\right]}$$

alors la convergence de I_1 a lieu également ; on peut donc calculer M, qui est défini par les moments du premier et du second ordre de la loi (16.196) ; le calcul montre que le moment du premier ordre est égal à $-H^{-1}.a$ et que les composantes du tenseur *variance* (16.196) sont égales aux éléments de la matrice H^{-1} ; le moment du second ordre s'en déduit immédiatement.

La formule (16.200) donne l'*entropie* :

$$\boxed{s = \frac{n}{2}\log(2\pi e) - \frac{1}{2}\log(\text{dét}(H))} \; ;$$

([1]) Voir *Calcul linéaire*, tome II.
([2]) C'est-à-dire en recherchant l'*image* de la loi par l'application $x \mapsto x^*$.



http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Example of Multivariate Gaussian Law (real case)

Multivariate Gaussian law parameterized by moments

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2}} e^{-\frac{1}{2}(z-m)^T R^{-1}(z-m)}$$

$$\frac{1}{2}(z-m)^T R^{-1}(z-m) = \frac{1}{2}\left[z^T R^{-1} z - m^T R^{-1} z - z^T R^{-1} m + m^T R^{-1} m\right]$$

$$= \frac{1}{2} z^T R^{-1} z - m^T R^{-1} z + \frac{1}{2} m^T R^{-1} m$$

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2} e^{\frac{1}{2} m^T R^{-1} m}} e^{-\left[-m^T R^{-1} z + \frac{1}{2} z^T R^{-1} z\right]} = \frac{1}{Z} e^{-\langle \xi, \beta \rangle}$$

$$\xi = \begin{bmatrix} z \\ zz^T \end{bmatrix} \text{ and } \beta = \begin{bmatrix} -R^{-1} m \\ \frac{1}{2} R^{-1} \end{bmatrix} = \begin{bmatrix} a \\ H \end{bmatrix} \text{ with } \langle \xi, \beta \rangle = a^T z + z^T H z = Tr\left[z a^T + H^T z z^T\right]$$

Gaussian Density is a 1st order Maximum Entropy Density !

THALES

Information Geometry for Multivariate Gaussian Density

- For multivariate gaussian density of meam m and covariance matrix R , classical parameterization is given by:

$$p_{m,R}(z) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2}} e^{-\frac{1}{2}(z-m)^T R^{-1}(z-m)}$$

- New parameterization by Information Geometry as Gibbs density:

$$p_{\bar{\xi}}(\xi) = \frac{1}{\int\limits_{\Omega^*} e^{-\langle \xi,\beta\rangle}.d\xi} e^{-\langle \xi,\beta\rangle} \quad \text{with} \quad \xi = \begin{bmatrix} z \\ zz^T \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} = \begin{bmatrix} a \\ H \end{bmatrix}$$

Duality bracket given by $\langle \xi,\beta\rangle = a^T z + z^T Hz = Tr\left[za^T + H^T zz^T \right]$

$$\log\left(\int\limits_{\Omega^*} e^{-\langle \xi,\beta\rangle}.d\xi\right) = \log(Z) = n\log(2\pi) + \frac{1}{2}\log\det(R) + \frac{1}{2}m^T R^{-1}m$$

Information Geometry for Multivariate Gaussian Density

Massieu characteristic function:

$$\psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \xi, \beta \rangle}.d\xi \quad \text{and} \quad \beta = \begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} = \begin{bmatrix} a \\ H \end{bmatrix}$$

$$\Phi(\beta) = -\log \psi_\Omega(\beta) = \frac{1}{2}\left[-Tr\left[H^{-1}aa^T \right] + \log\left[(2)^n \det H \right] - n\log(2\pi) \right]$$

Deriving relation providing moment:

$$\frac{\partial \Phi(\beta)}{\partial \beta} = \int_{\Omega^*} \xi . p_{\bar{\xi}}(\xi) d\xi = \bar{\xi} = \begin{bmatrix} E[z] \\ E\left[zz^T\right] \end{bmatrix} = \begin{bmatrix} m \\ R + mm^T \end{bmatrix}$$

$$\text{with} \quad \xi = \begin{bmatrix} z \\ zz^T \end{bmatrix} \quad \text{and} \quad R = E\left[(z-m)(z-m)^T \right] = E\left[zz^T \right] - mm^T$$

Information Geometry for Multivariate Gaussian Density

(Shannon) Entropy, Legendre transform of Massieu characteristic function

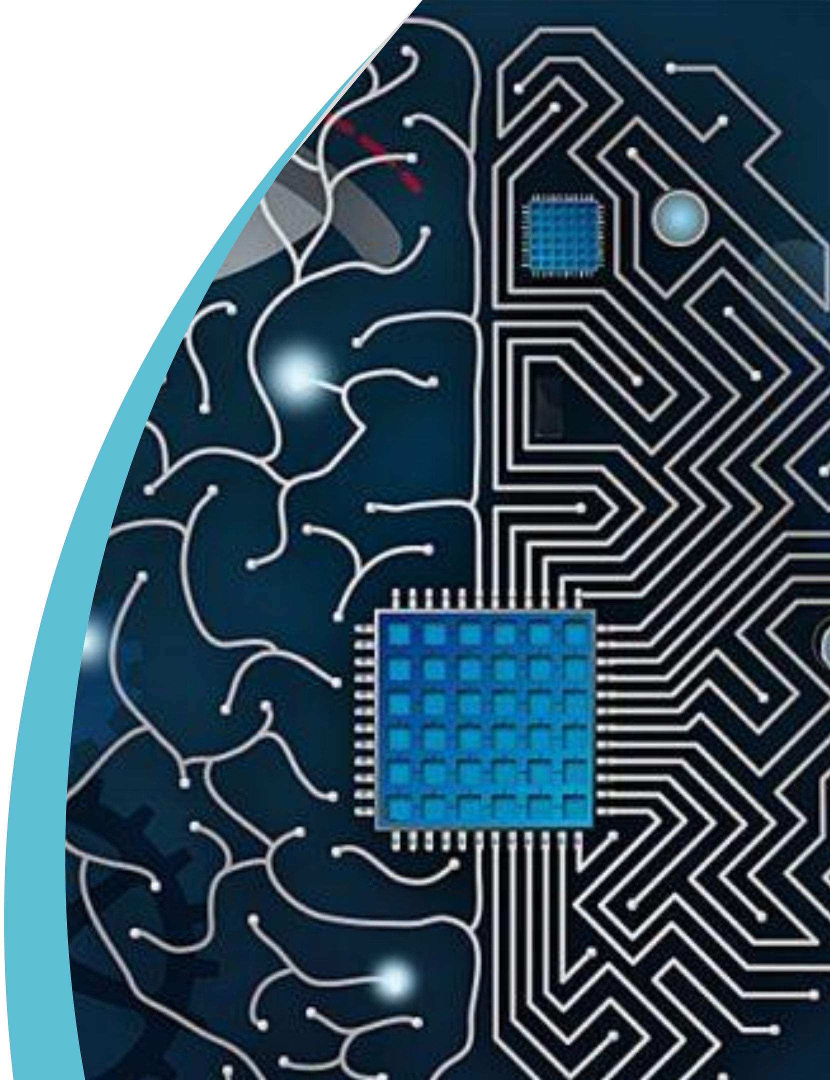
$$S\left(\bar{\xi}\right)=\left\langle \bar{\xi},\beta\right\rangle-\Phi\left(\beta\right) \quad with \quad \frac{\partial S\left(\bar{\xi}\right)}{\partial\bar{\xi}}=\beta=\begin{bmatrix} a \\ H \end{bmatrix}=\begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix}$$

$$S\left(\bar{\xi}\right)=-\int_{\Omega^*}\frac{e^{-\langle\xi,\beta\rangle}}{\int_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi}\log\frac{e^{-\langle\xi,\beta\rangle}}{\int_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi}.d\xi=-\int_{\Omega^*}p_{\bar{\xi}}(\xi)\log p_{\bar{\xi}}(\xi).d\xi$$

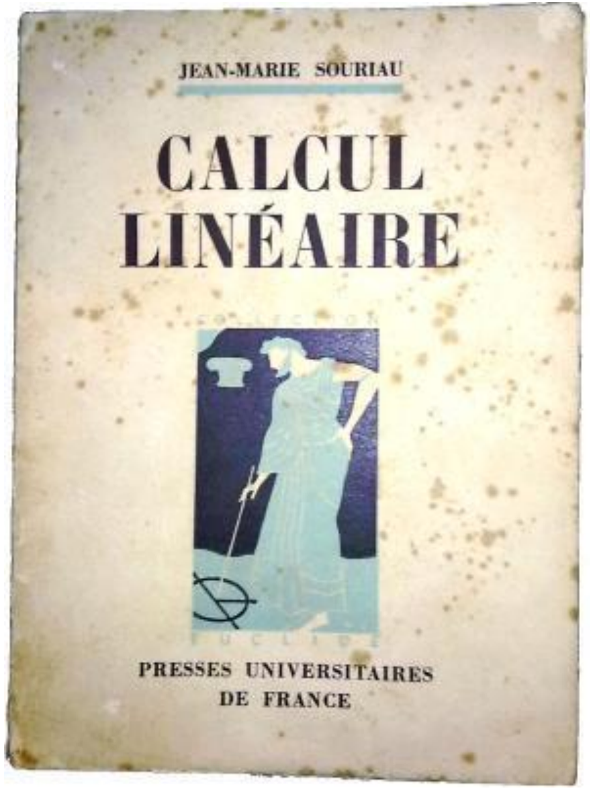
(Shannon) Entropy with new parameterization:

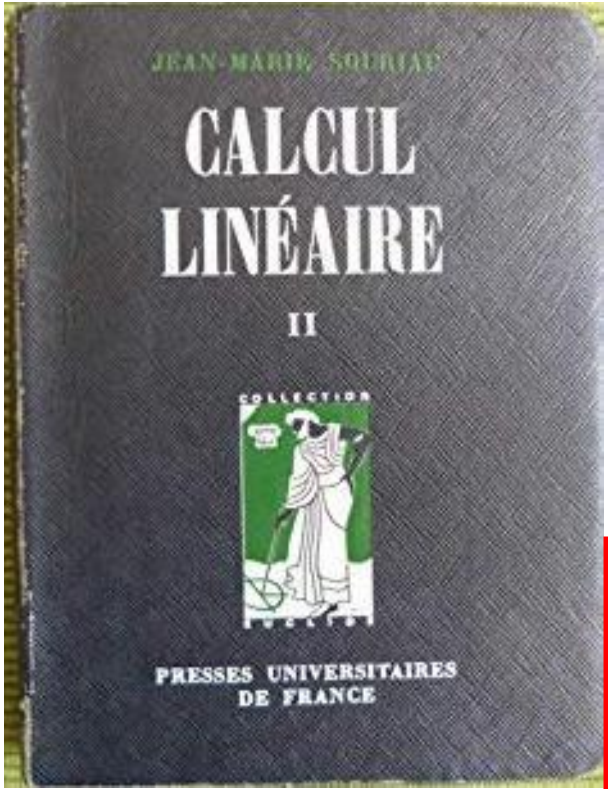
$$S\left(\bar{\xi}\right)=\frac{1}{2}\left[\log(2)^n\det\left[H^{-1}\right]+n\log\left(2\pi.e\right)\right]=\frac{1}{2}\left[\log\det\left[R\right]+n\log\left(2\pi.e\right)\right]$$

Souriau Algorithm for Exponential Map



Souriau Book on « Calcul Linéaire » & Leverrier-Souriau Algorithm





$$P(\lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} k_i \lambda^{n-i}$$

$$k_0 = 1 \quad \text{et} \quad B_0 = I$$

$$\begin{cases} A_i = B_{i-1}A \quad , \quad k_i = -\dfrac{1}{i} tr(A_i), \quad i = 1,\dots,n-1 \\ B_i = A_i + k_i I \quad \text{ou} \quad B_i = B_{i-1}A - \dfrac{1}{i} tr(B_{i-1}A) I \end{cases}$$

$$A_n = B_{n-1}A \quad \text{et} \quad k_n = -\frac{1}{n} tr(A_n)$$

Souriau, J.-M..:Une méthode pour la décomposition spectrale et l'inversion des matrices. Comptes-Rendus hebdomadaires des séances de l'Académie des Sciences 227 (2), 1010–1011, Gauthier-Villars, Paris (1948).

Souriau Algorithm for Characteristic Polynomial Computation

Souriau Algorithm (1948)

$$P(\lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} k_i \lambda^{n-i} \qquad Q(\lambda) = adj(\lambda I - A) = \sum_{i=0}^{n-1} \lambda^{n-i-1} B_i$$

$$k_0 = 1 \quad \text{and} \quad B_0 = I$$

$$\begin{cases} A_i = B_{i-1} A \quad , \quad k_i = -\frac{1}{i} tr(A_i), \quad i = 1, \ldots, n-1 \\ B_i = A_i + k_i I \quad \text{or} \quad B_i = B_{i-1} A - \frac{1}{i} tr(B_{i-1} A) I \end{cases}$$

$$A_n = B_{n-1} A \quad \text{and} \quad k_n = -\frac{1}{n} tr(A_n)$$

Souriau Extension Algorithm for Exponential Map

$$[\lambda I - A]^{-1} = \frac{Q(\lambda)}{P(\lambda)} \Leftrightarrow [\lambda I - A]Q(\lambda) = P(\lambda)I \qquad \left[I\frac{d}{dt} - A\right]Q\left(\frac{d}{dt}\right) = P\left(\frac{d}{dt}\right)I \qquad P\left(\frac{d}{dt}\right)\gamma = 0$$

$$\Phi = Q\left(\frac{d}{dt}\right)\gamma \qquad \frac{d\Phi}{dt} = A\Phi(t)$$

$$1)\begin{cases} B_0 = I \ \text{ and } \ B_i = B_{i-1}A - \dfrac{tr\left(B_{i-1}A\right)}{i}I \\ k_0 = 1 \ , \ k_i = -\dfrac{tr\left(B_{i-1}A\right)}{i} \quad i = 1,\dots,n \end{cases}$$

$$2)\begin{cases} \gamma \ \text{ integrated on } [0,h] \text{ such that} \\ k_0\gamma^{(n)} + k_1\gamma^{(n-1)} + \dots + k_{n-1}\gamma^{(1)} + k_n\gamma = 0 \\ \text{with } \ \gamma(0) = \dots = \gamma^{(n-2)} = 0 \ \text{ and } \ \gamma^{(n-1)}(0) = 1 \end{cases}$$

$$3) \text{ Computation of } \Phi(t) = e^{tA} = \sum_{i=0}^{n-1}\gamma^{(n-i-1)}(t)B_i \ \text{ on } \ [0,h]$$

$$4) \text{ Extension of Computation on } [0, ph] \text{ by } \Phi(pt) = (\Phi(t))^p$$

$$5) \ X(t) = \Phi(t)X_0 \ \text{ with } \ X_0 = X(0)$$

Good Candidate due to its high parallelization capability for Exponential Map Computation for « Lie Group Machine Learning »

THALES

Souriau algorithm to recover Lie Group Rodrigue's formula

$$SO(3)=\left\{R / R^{-1}=R^{T}\right\} \qquad \omega_{\times}=\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}=\omega_1 L_1+\omega_2 L_2+\omega_3 L_3 \in \mathfrak{so}(3), \omega=\left(\omega_1, \omega_2, \omega_3\right) \in \mathbb{R}^3$$

$$e^{\omega_{\times}}=\gamma^{(2)} B_0+\gamma^{(1)} B_1+\gamma B_2$$

$$B_0=I \text{ and } k_0=1$$

$$B_1=I.\omega_{\times}-\frac{Tr\left(I.\omega_{\times}\right)}{1} I=\omega_{\times} \text{ and } k_1=-\frac{Tr\left(I\omega_{\times}\right)}{1}=0$$

$$\Rightarrow \quad e^{\omega_{\times}}=\gamma^{(2)} I+\gamma^{(1)} \omega_{\times}+\gamma \omega \otimes \omega^{T}$$

$$B_2=B_1.\omega_{\times}-\frac{Tr\left(\omega_{\times}.\omega_{\times}\right)}{2} I=\omega_{\times}.\omega_{\times}+\|\omega\|^2 I \text{ and } k_2=-\frac{Tr\left(\omega_{\times}.\omega_{\times}\right)}{2}=\|\omega\|^2$$

$$B_2=\omega_{\times}.\omega_{\times}+\|\omega\|^2 I=\omega \otimes \omega^{T} \qquad k_3=0$$

$$\gamma^{(3)}(t)+\|\omega\|^2 \gamma^{(1)}(t)=0 \text{ with } \gamma^{(2)}(0)=1, \gamma^{(1)}(0)=0, \gamma(0)=0$$

$$\gamma^{(1)}(t)=\frac{1}{\|\omega\|} \sin\left(\|\omega\| t\right) \text{ and } \gamma(t)=\frac{1}{\|\omega\|^2}\left(1-\cos\left(\|\omega\| t\right)\right)$$

$$\omega \otimes \omega^{T}=\omega_{\times}.\omega_{\times}+\|\omega\|^2 I \quad \Rightarrow$$

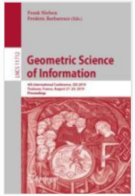
$$e^{t.\omega_{\times}}=I+\frac{1}{\|\omega\|} \sin\left(\|\omega\| t\right) \omega_{\times}+\frac{1-\cos\left(\|\omega\| t\right)}{\|\omega\|^2} \omega_{\times}^2$$

Reference

Souriau Exponential Map Algorithm for Machine Learning on Matrix Lie Groups

- https://link.springer.com/chapter/10.1007%2F978-3-030-26980-7_10

Springer Link



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Souriau Exponential Map Algorithm for Machine Learning on Matrix Lie Groups

Authors   Authors and affiliations

Frédéric Barbaresco

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Natural Exponential Families Invariant by a Group: Muriel Casalis & Gérard Letac

Seminal work of Muriel Casalis (Institut Mathématique de Toulouse)

Muriel Casalis PhD at Paul Sabatier Toulouse University supervised by Gérard Letac

Reference of Muriel Casalis

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Familles Exponentielles Naturelles sur $\mathbb{R}^d$ Invariantes par un Groupe

Muriel Casalis

Laboratoire de Statistique, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex, France

Résumé

La caractérisation des familles exponentielles naturelles de $\mathbb{R}^d$ préservées par un groupe d'affinités donné est faite dans trois cas: celui d'un groupe compact, en particulier du groupe des rotations, celui du groupe hyperbolique et enfin celui d'un groupe quelconque de translations. La démarche adoptée consiste à traduire la propriété d'invariance de la famille par une propriété portant sur les mesures qui l'engendrent puis à rechercher ces dernières en conséquence.

1 Introduction

Il est courant en statistique d'envisager un modèle $(\Omega, \mathscr{A}, (P_\theta)_{\theta\in\Theta})$ tel qu'il existe un groupe G de permutations de Ω préservant globalement la famille de probabilités $F = \{P_\theta, \theta \in \Theta\}$, c'est-à-dire que pour tout (θ, g) de $\Theta \times G$, l'image $g(P_\theta)$ de P_θ par g est encore dans F (Barndorff-Nielsen parle alors de modèle de transformations). On pourra consulter Barndorff-Nielsen et al. (1982) et plus récemment le livre de Barndorff-Nielsen (1988).

Un exemple célèbre est celui des distributions de Fisher–Von-Mises pour lequel Ω est la sphère unité de l'espace euclidien E,

$$P_\theta(dx) = L(\theta)^{-1} \exp\langle\theta, x\rangle \sigma(dx),$$

σ désignant la probabilité uniforme sur Ω et $L(\theta)$ le coefficient de normalisation, et pour lequel G est le groupe des rotations $0(E)$ de E.

Dans cet exemple, $\{P_\theta, \theta \in \Theta\}$ est une famille exponentielle naturelle au sens suivant.

Soit E un espace vectoriel de dimension finie, E^* son dual et si (θ, x) est dans $E^* \times E$, $\langle\theta, x\rangle$ désigne le crochet de dualité; soit, de plus, μ une mesure de Radon positive sur E; on note alors L_μ la transformée de Laplace de μ définie par:

$$L_\mu : E^* \to [0, \infty] : \theta \to \int_E \exp\langle\theta, x\rangle \mu(dx);$$

D_μ est l'ensemble $\{\theta \in E^*, L_\mu(\theta) < \infty\}$, $\Theta(\mu)$ son intérieur et k_μ la fonction définie sur $\Theta(\mu)$ par:

$$k_\mu(\theta) = \operatorname{Log} L_\mu(\theta). \tag{1.1}$$

On désigne aussi par $\mathscr{M}(E)$ l'ensemble des mesures de Radon μ positives telles que:

(i) μ n'est pas concentrée sur un sous-espace affine strict de E;
(ii) $\Theta(\mu)$ est non vide.

N° d'ordre 679

THESE

présentée à

L'UNIVERSITE PAUL SABATIER DE TOULOUSE (SCIENCES)

pour obtenir

DOCTORAT DE L'UNIVERSITE PAUL SABATIER

Spécialité : MATHEMATIQUES APPLIQUEES

par

Muriel BONNEFOY - CASALIS

FAMILLES EXPONENTIELLES NATURELLES

INVARIANTES PAR UN GROUPE

Soutenue le 11 Juin 1990, devant la Commission d'Examen :

MM.	H. CAUSSINUS	Professeur à l'Université Paul Sabatier
	D. BAKRY	Professeur à l'Université Paul Sabatier
	J. FARAUT	Professeur à l'Université PARIS VI
	Y. GUIVARC'H	Professeur à l'Université PARIS VI
	G. LETAC	Professeur à l'Université Paul Sabatier

Laboratoire de Statistique et Probabilités
UNIVERSITE PAUL SABATIER

NEF (Natural Exponential Families): Letac & Casalis

- Let E a vector space of finite size, E^* its dual. $\langle \theta, x \rangle$ Duality braket with $(\theta, x) \in E^* \times E$. μ Positive Radon measure on E, Laplace transform is :

$$L_\mu : E^* \to [0, \infty] \text{ with } \theta \mapsto L_\mu(\theta) = \int_E e^{\langle \theta, x \rangle} \mu(dx)$$

- Transformation $k_\mu(\theta)$ defined on $\Theta(u)$ interior of $D_\mu = \{\theta \in E^*, L_\mu < \infty\}$

$$k_\mu(\theta) = \log L_\mu(\theta)$$

- Natural exponential families are given by:

$$F(\mu) = \left\{ P(\theta, \mu)(dx) = e^{\langle \theta, x \rangle - k_\mu(\theta)} \mu(dx), \theta \in \Theta(\mu) \right\}$$

- Injective function (domian of means): $k'_\mu(\theta) = \int_E x P(\theta, \mu) \mu(dx)$

- And the inverse function: $\psi_\mu : M_F \to \Theta(\mu)$ with $M_F = \operatorname{Im}(k'_\mu(\Theta(\mu)))$

- Covariance operator: $V_F(m) = k''_\mu(\psi_\mu(m)) = (\psi'_\mu(m))^{-1}, m \in M_F$

NEF (Natural Exponential Families): Letac & Casalis

- Measure generetad by a familly F :

$$F(\mu)=F(\mu')\Leftrightarrow \exists(a,b)\in E^{*}\times R, \text{such that } \mu'(dx)=e^{\langle a,x\rangle+b}\mu(dx)$$

- Let F an exponential family of E generated by μ and $\varphi : x \mapsto g_{\varphi}x+v_{\varphi}$

  with $g_{\varphi}\in GL(E)$ automorphisms of E and $v_{\varphi}\in E$, then the familly

  $\varphi(F)=\{\varphi(P(\theta,\mu)),\theta\in\Theta(\mu)\}$ is an exponential familly of E

  generated by $\varphi(\mu)$

- <u>Definition:</u> An exponential familly F is invariant by a group G (affine group of E), if $\forall\varphi\in G, \varphi(F)=F$: $\forall\mu, F(\varphi(\mu))=F(\mu)$

  (the contrary could be false)

NEF (Natural Exponential Families): Letac & Casalis

- Theorem (Casalis): Let $F = F(\mu)$ an exponential familly of E and G affine group of E, then F is invariant by G if and only:

$$\exists a : G \to E^{*}, \exists b : G \to R, \text{such that}:$$

$$\forall (\varphi, \varphi') \in G^2, \begin{cases} a(\varphi\varphi') = {}^t g_\varphi^{-1} a(\varphi') + a(\varphi) \\ b(\varphi\varphi') = b(\varphi) + b(\varphi') - \left\langle a(\varphi'), g_\varphi^{-1} v_\varphi \right\rangle \end{cases}$$

$$\forall \varphi \in G, \varphi(\mu)(dx) = e^{\langle a(\varphi), x\rangle + b(\varphi)} \mu(dx)$$

- When G is a linear subgroup, b is a character of G, a could be obtained by the help of Cohomology of Lie groups .

NEF (Natural Exponential Families): Letac & Casalis

- If we define action of G on E^* by: $g.x={}^t g^{-1}x, g \in G, x \in E^*$

  we can verify that: $a(g_1 g_2) = g_1.a(g_2) + a(g_1)$

- the action a is an inhomogeneous 1-cocycle: $\forall n > 0$, let the set of all functions from G^n to E^*, $\Im(G^n, E^*)$ called inhomogenesous n-cochains, then we can define the operators: $d^n : \Im(G^n, E^*) \to \Im(G^{n+1}, E^*)$

$$d^n F(g_1, \cdots, g_{n+1}) = g_1.F(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i F(g_1, g_2, \cdots, g_i g_{i+1}, \cdots, g_n)$$

$$+(-1)^{n+1} F(g_1, g_2, \cdots, g_n)$$

NEF (Natural Exponential Families): Letac & Casalis

- Let $Z^n(G,E^*) = Ker(d^n)$, $B(G,E^*) = \mathrm{Im}(d^{n-1})$, with Z^n inhomogneous n-cocycles , the quotient $H^n(G,E^*) = Z^n(G,E^*)/B^n(G,E^*)$ is the **Cohomology Group** of G with value in E^*. We have:

$$d^0 : E^* \to \Im(G,E^*) \qquad Z^0 = \{x \in E^*; g.x = x, \forall g \in G\}$$

$$x \mapsto (g \mapsto g.x - x)$$

$$d^1 : \Im(G,E^*) \to \Im(G^2,E^*)$$

$$F \mapsto d^1F \quad , \quad d^1F(g_1,g_2) = g_1.F(g_2) - F(g_1g_2) + F(g_1)$$

$$Z^1 = \{F \in \Im(G,E^*); F(g_1g_2) = g_1.F(g_2) + F(g_1), \forall (g_1,g_2) \in G^2\}$$

$$B^1 = \{F \in \Im(G,E^*); \exists x \in E^*, F(g) = g.x - x\}$$

NEF (Natural Exponential Families): Letac & Casalis

- When **the Cohomology Group** $H^1(G,E^*)=0$ then $Z^1(G,E^*)=B^1(G,E^*)$

$$\Rightarrow \exists c \in E^*, \text{such that } \forall g \in G, a(g)=\left(I_d - {}^t g^{-1}\right)c$$

Then if $F=F(\mu)$ is an exponential familly invariant by G , μ verifies

$$\forall g \in G, g(\mu)(dx)=e^{\langle c,x\rangle - \langle c, g^{-1}x\rangle + b(g)}\mu(dx)$$

$$\forall g \in G, g\left(e^{\langle c,x\rangle}\mu(dx)\right)=e^{b(g)}e^{\langle c,x\rangle}\mu(dx) \text{ with } \mu_0(dx)=e^{\langle c,x\rangle}\mu(dx)$$

- For all compact Group, $H^1(G,E^*)=0$ and we can express a

$$A: G \rightarrow GA(E) \qquad \forall (g,g') \in G^2, A_{gg'}=A_g A_{g'}$$

$$g \mapsto A_g \;,\; A_g(\theta)={}^t g^{-1}\theta + a(g) \qquad A(G) \text{ compact sub-group of } GA(E)$$

$$\exists \text{fixed point} \Rightarrow \forall g \in G, A_g(c)={}^t g^{-1}c + a(g) = c \Rightarrow a(g)=\left(I_d - {}^t g^{-1}\right)c$$

THALES

Bargmann Parameterization of SU(1,1) Lie Group

Bargmann parameterization of SU(1,1)

- $SU(1,1)$ is isomorphic to $SL(2,R)=Sp(2,R)$ through the complex unitary matrix W :

$$SL(2,R)=\left\{ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} / \det g = ad-bc=1 \right\}$$

$$Sp(2,R)=\left\{ g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} / gJg^T = J, J=\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$W=\frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{-1} & \omega^{-1} \\ -\omega & \omega \end{pmatrix}=\left(W^{+}\right)^{-1} \quad \text{with} \quad \omega=e^{i\pi/4}=\frac{1}{\sqrt{2}}(1+i)$$

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Bargmann parameterization of SU(1,1)

- If we observe that $W^{-1}JW=-iM$, the isomorphism is given explicitely by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g(u) = WuW^{-1} = \begin{pmatrix} \mathrm{Re}(\alpha+\beta) & -\mathrm{Im}(\alpha-\beta) \\ \mathrm{Im}(\alpha+\beta) & \mathrm{Re}(\alpha-\beta) \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = u(g) = W^{-1}gW = \frac{1}{2}\begin{pmatrix} (a+d)-i(b-c) & (a-d)+i(b+c) \\ (a-d)-i(b+c) & (a+d)+i(b-c) \end{pmatrix}$$

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^{-1} & \omega^{-1} \\ -\omega & \omega \end{pmatrix} = \left(W^+\right)^{-1} \quad \text{with} \quad \omega = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$$

Bargmann parameterization of SU(1,1)

- We can also make also a link with $SO(2,1)$ of "1+2" pseudo-orthogonal matrices:

$$SO(2,1)=\left\{\Gamma\in GL(3,3)/\det(\Gamma)=1,\Gamma K\Gamma^{T}=\Gamma,K=\begin{pmatrix}+1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\right\}$$

$$\Gamma(g)=\begin{pmatrix}\frac{1}{2}\left(a^2+b^2+c^2+d^2\right)&\frac{1}{2}\left(a^2-b^2+c^2-d^2\right)&-cd-ab\\\frac{1}{2}\left(a^2+b^2-c^2-d^2\right)&\frac{1}{2}\left(a^2-b^2-c^2+d^2\right)&cd-ab\\-bd-ac&bd-ac&ad+bc\end{pmatrix}$$

with $\Gamma(g_1)\Gamma(g_2)=\Gamma(g_1g_2),\Gamma(I)=I,\Gamma(g^{-1})=\Gamma(g)^{-1}$

Bargmann parameterization of SU(1,1)

- The matrix $SO(2,1)$ corresponds to any $SU(1,1)$:

$$\Gamma(u)=\begin{pmatrix} |\alpha|^2+|\beta|^2 & 2\operatorname{Re}\alpha\beta^* & 2\operatorname{Im}\alpha\beta^* \\ 2\operatorname{Re}\alpha\beta & \operatorname{Re}\left(\alpha^2+\beta^2\right) & \operatorname{Im}\left(\alpha^2-\beta^2\right) \\ -2\operatorname{Im}\alpha\beta & -\operatorname{Im}\left(\alpha^2+\beta^2\right) & \operatorname{Re}\left(\alpha^2-\beta^2\right) \end{pmatrix}$$

$$\alpha=\pm\sqrt{\frac{1}{2}\left(\Gamma_{11}+\Gamma_{12}\right)+i\left(\Gamma_{12}-\Gamma_{21}\right)},\quad \beta=\frac{1}{2\alpha}\left(\Gamma_{10}-i\Gamma_{20}\right)$$

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Bargmann parameterization of SU(1,1)

- The properties of connectivity of $Sp(2,R)$ is described by its isomorphy with $SU(1,1)$
- Using unimodular condition:

$$|\alpha|^2 - |\beta|^2 = 1 \Rightarrow \alpha_R^2 + \alpha_I^2 - \beta_R^2 = 1 + \beta_I^2 \geq 1$$

$$\text{with} \quad \alpha = \alpha_R + i\alpha_I \ \text{ and } \beta = \beta_R + i\beta_I$$

- If β_I is fixed, $(\alpha_R, \alpha_I, \beta_R)$ are constrained to define a one-sheeted revolution hyperboloid, with its circular waist in the α plane.

Bargmann parameterization of SU(1,1)

- To $SU(1,1)$, we can associate the simply-connected universal covering group, using the maximal compact subgroup $U(1)$ and corresponding to the Iwasawa decomposition (factorization of a noncompact semisimple group into its maximal compact subgroup times a solvable subgroup).

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{i\omega} \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda \end{pmatrix} \text{ with } \begin{cases} \omega = \arg\alpha = \dfrac{1}{2} i \ln\left(\alpha^* \alpha^{-1}\right) \\ \lambda = |\alpha| > 0 \\ \mu = e^{-i\omega}\beta = \sqrt{\dfrac{\alpha^*}{\alpha}}\beta \end{cases}$$

$$\beta = e^{i\omega}\mu, |\alpha|^2 - |\beta|^2 = \lambda^2 - |\mu|^2 = 1 \text{ so } |\mu| < \lambda$$

Bargmann parameterization of SU(1,1)

- Bargmann has generalized this parameterization for $Sp(2N,R)$, more convenient but difficult to generalize to N dimensions.
- For $SU(1,1)$, Bargmann has used (ω,γ) :

$$\gamma=\frac{\mu}{\lambda}=\frac{\beta}{\alpha}\ \left(|\gamma|<1\right)\ ,\ \lambda=\frac{1}{\sqrt{1-|\gamma|^2}},\ \mu=\frac{\gamma}{\sqrt{1-|\gamma|^2}}$$

- For $SL(2,R)=Sp(2,R)$, the Bargman, parameterization is given by this decom-position of a non-singular matrix into the product of an orthogonal and a positive definite symmetric matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}\begin{pmatrix} \lambda+\operatorname{Re}\mu & \operatorname{Im}\mu \\ \operatorname{Im}\mu & \lambda-\operatorname{Re}\mu \end{pmatrix}$$

$$\omega=\arg\left[(a+d)-i(b-c)\right],\ \mu=e^{-i\omega}\left[(a-d)+i(b+c)\right]$$

- $SU(1,1)$ and $SL(2,R)=Sp(2,R)$ are described when is counted modulo 2π

$$\omega=\omega(\operatorname{mod}2\pi)$$