Non-linear Embeddings in Hilbert Simplex Geometry

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Contributions:
- Simple proof of monotonicity of Hilbert distance
- Connection of Hilbert distance with Aitchison distance
- Differentiable approximation of Hilbert distance
- Application to non-linear embedding: experimentally fast, robust, and competitive

Open bounded convex $\Omega$ of $\mathbb{R}^d$:

Hilbert simplex distance:

$$\Delta_d := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d_+ : \sum_{i=1}^d x_i = 1 \}$$

$$\rho_{\text{FD}}(p, q) = \log \max_{i \in \{1, \ldots, d\}} \frac{p_i}{q_i} \frac{q_i}{p_i}$$

$$\rho_{\text{HG}}(p, q) = \log \max_{i \in \{1, \ldots, d\}} \frac{p_i}{q_i} \frac{q_i}{p_i}$$

Aitchison distance:

$$\rho_{\text{Aitch}}(p, q) := \sqrt{\sum_{i=1}^d \left( \log \frac{p_i}{q_i(\log q_i)} - \log \frac{q_i}{p_i(\log p_i)} \right)}$$

geometric mean:

$$G(p) = \left( \prod_{i=1}^d p_i \right)^{\frac{1}{d}} = \exp \left( \frac{1}{d} \sum_{i=1}^d \log p_i \right)$$

Hilbert metric distance: Symmetrize Funk distance

$$\rho_{\text{HG}}(p, q) := \rho_{\text{FD}}(p, q) + \rho_{\text{FD}}(q, p) = \begin{cases} \log \frac{\|p - q\|}{\|q - p\|}, & p \neq q, \\ 0, & p = q. \end{cases}$$

Straight line segments = geodesics but geodesics not unique:

$$\bar{q} = \text{ray}(p, q) \cap \partial \Omega$$

Differentiable approximation:

$$\hat{\rho}_{\text{SET}}(p, q) := \frac{1}{d} \log \left( \sum_i \left( \frac{p_i}{q_i} \right)^T \left( \sum_i \left( \frac{p_i}{q_i} \right) \right)^T \right)$$

$$\lim_{T \to \infty} \hat{\rho}_{\text{SET}}^T(p, q) = \rho(p, q)$$

Loss functions:
Finite average KLD
+ Adam optimizer

See experiments in arxiv:2203.11434