

A closed-form expression of geodesics in the Klein model of hyperbolic geometry

Frank Nielsen
Sony Computer Science Laboratories Inc.
Tokyo, Japan

April 2021

Hyperbolic geometry [3] is more and more often used in machine learning and computer vision, specially to embed and process hierarchical structures (e.g., [9, 8]). The five main models of hyperbolic geometry [3] are the Poincaré upper space model, the Poincaré ball model, the Beltrami hemisphere model, the Lorentz hyperboloid model, and the Klein ball model. These models yield metric spaces (\mathbb{D}, d) where \mathbb{D} denotes the domain of the model and $d(\cdot, \cdot)$ denotes the hyperbolic distance, a metric distance. These metric spaces are said geodesic because there exists a map $\gamma(p, q; \alpha)$ such that

$$d(\gamma(p, q; s), \gamma(p, q; t)) = |s - t| d(p, q), \quad \forall p, q \in \mathbb{D}, \quad \forall s, t \in [0, 1].$$

A closed-form expression of the geodesics in the hyperbolic Poincaré ball model can be expressed using Möbius operations in the Möbius ball gyrovector space [2, 8].

In this note, we report a closed-form expression of the geodesics in the Klein ball model of arbitrary dimension. Although the Klein ball model (K) is not conformal (except at the origin), the traces of geodesics $\Gamma_K(p, q) = \{(1 - \alpha)p + \alpha q : \alpha \in [0, 1]\}$ (called pregeodesics) are straight line segments making it convenient for robust geometric computing (e.g., Klein hyperbolic Voronoi diagram [5, 6]). Once a structure is computed in the Klein model, it can be converted into the other models (e.g., [7]).

Let $\mathbb{B}_n = \{x \in \mathbb{R}^n : x^\top x < 1\}$ be the n -dimensional unit open ball centered at the origin. The Klein distance $d_K(p, q)$ between point p and q in \mathbb{B}_n (hyperbolic geometry with curvature $\kappa = -1$) is

$$d_K(p, q) = \operatorname{arccosh} \left(\frac{1 - p^\top q}{\sqrt{(1 - p^\top p)} \sqrt{(1 - q^\top q)}} \right).$$

Thus we seek a parameterization

$$\gamma_K(p, q; \alpha) = (1 - u(\alpha))p + u(\alpha)q \tag{1}$$

so that

$$d_K(\gamma_K(p, q; s), \gamma_K(p, q; t)) = |s - t| d_K(p, q).$$

In particular, when $s = 0$ and $t = \alpha$, we shall have

$$d_K(p, (1 - u(\alpha))p + u(\alpha)q) = \alpha d_K(p, q).$$

This latter equation amounts to solve for $u(\alpha)$ in the equation:

$$\frac{a - bu(\alpha)}{\sqrt{a(a - 2bu(\alpha) + cu(\alpha)^2)}} - d(\alpha) = 0,$$

where

$$\begin{aligned} a &= 1 - p^\top p, \\ b &= p^\top (q - p), \\ c &= (q - p)^\top (q - p), \\ d(\alpha) &= \cosh(\alpha d_K(p, q)) \end{aligned}$$

Using symbolic calculations, we find the following solution:

$$u(\alpha) = \frac{ad(\alpha)\sqrt{(ac + b^2)(d(\alpha)^2 - 1)} + ab(1 - d(\alpha)^2)}{acd(\alpha)^2 + b^2}. \quad (2)$$

Thus we get in closed-form the Klein geodesics γ_K (albeit a large formula) such that

$$d_K(\gamma(p, q; s), \gamma(p, q; t)) = |s - t| d_K(p, q).$$

The snippet code below implements in MAXIMA¹ the geodesics in the Klein model with a test set.

```
dKlein(p, q) := acosh((1 - p . q) / (sqrt((1 - p . p) * (1 - q . q))));

u(p, q, alpha) := ((1 - p . p) * cosh(alpha * dKlein(p, q)) * sqrt(((1 - p . p) * ((q - p) . (q - p)) + (p . (q - p)) ** 2) * (cosh(alpha * dKlein(p, q)) ** 2 - 1)) + (1 - p . p) * (p . (q - p)) * (1 - cosh(alpha * dKlein(p, q)) ** 2)) / ((1 - p . p) * ((q - p) . (q - p)) * cosh(alpha * dKlein(p, q)) ** 2 + (p . (q - p)) ** 2);

gammaKlein(p, q, alpha) := (1 - u(p, q, alpha)) * p + u(p, q, alpha) * q;

/* Test */
p: [0.5, 0.2];
q: [0.1, -0.3];

/* 1st test for Klein geodesics */
alpha: random(1.0);
alpha * dKlein(p, q);
dKlein(p, gammaKlein(p, q, alpha));

/* 2nd test for Klein geodesics */
s: random(1.0);
t: random(1.0);
dKlein(gammaKlein(p, q, s), gammaKlein(p, q, t));
abs(s - t) * dKlein(p, q);
```

¹<https://maxima.sourceforge.io/>

To illustrate the use of these Klein geodesics, consider computing the smallest enclosing ball of a finite point set $\mathcal{P} = \{p_1, \dots, p_m\}$ in hyperbolic geometry [1]. The closed-form expression of the Klein geodesic, allows to bypass the use of hyperbolic translations from/to the ball origin as this was used in [4]. The algorithm for calculating an approximation of the hyperbolic smallest enclosing ball is:

- Initialize $c_1 = p_1$
- Repeat t times: Let $c_{i+1} = \gamma_K \left(c_i, p_{f_i}, \frac{1}{i+1} \right)$ where p_{f_i} is the farthest point of \mathcal{P} to c_i . That is, we have $f_i = \arg \max_{j \in \{1, \dots, m\}} d_K(c_i, p_j)$.

The algorithm is proven to converge in [1] (i.e., $\lim_{t \rightarrow \infty} c_t = \arg \min_c \max_{i \in \{1, \dots, m\}} d_K(p_i, c)$) since the hyperbolic geometry is a Hadamard space.

Additional material is available online at <https://franknielsen.github.io/KleinGeodesics/>

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